ONE-PARAMETER GENERALIZATIONS OF ROGERS-RAMANUJAN TYPE IDENTITIES

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Abstract. Resorting to the recursions satisfied by the polynomials which converge to the right hand sides of the Rogers-Ramanujan type identities given by Sills [17] and a determinant method presented in [9], we obtain many new one-parameter generalizations of the Rogers-Ramanujan type identities, such as a generalization of the analytic versions of the first and second Göllnitz-Gordon partition identities, and generalizations of the first, second, and third Rogers-Selberg identities.

1. Introduction

In [7], by evaluating an integral involving q-Hermite polynomials in two different ways and equating the results, Garrett et al. found a generalization of the celebrated Rogers-Ramanujan identities:

$$\sum_{n=0}^{\infty}\frac{q^{n^2+mn}}{(q;q)_n}=\frac{(-1)^m q^{-\binom{m}{2}}E_{m-2}}{(q,q^4;q^5)_\infty}-\frac{(-1)^m q^{-\binom{m}{2}}D_{m-2}}{(q^2,q^3;q^5)_\infty}, \tag{1.1}$$

where the Schur polynomials D_m and E_m are defined by

$$\begin{aligned} D_m &= D_{m-1}+q^m D_{m-2}, \qquad D_0=1,\ D_1=1+q,\\ E_m &= E_{m-1}+q^m E_{m-2}, \qquad E_0=1,\ E_1=1, \end{aligned}$$

and Schur [15] gave the limit

$$D_\infty=\frac{1}{(q,q^4;q^5)_\infty}, \qquad E_\infty=\frac{1}{(q^2,q^3;q^5)_\infty}.$$

It is obvious that we can get the following two Rogers-Ramanujan identities by letting $m=0$ and $m=1$ in (1.1), respectively.

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}=\frac{1}{(q,q^4;q^5)_\infty}, \tag{1.2}$$

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}=\frac{1}{(q^2,q^3;q^5)_\infty}. \tag{1.3}$$

Later, Andrews et al. [3] provided an alternative proof of (1.1) by using the extended Engel expansion. In [9], Ismail et al. used the theory of associated orthogonal polynomials to explain determinants that Schur introduced in 1917, and showed that Equation (1.1) can be obtained from the Rogers-Ramanujan identities (1.2) and (1.3). Furthermore, Andrews et al. [4] discussed Al-Salam/Ismail and Santos polynomials in the context of identities of (1.1) type.

The main purpose of this paper is to apply the determinant method which was presented in [9] to generalize the Rogers-Ramanujan type identities. In [17], Sills mainly focused on a method which

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was developed by Andrews [2, §9.2, p. 88] for discovering finite analogs of Rogers-Ramanujan type identities via q-difference equations. In the paper, he presented at least one finitization for each of the 130 identities in Slater's list [18], along with recursions satisfied by the polynomials which converge to the right hand sides of the Rogers-Ramanujan type identities. Resorting to these recursions and the determinant method, we obtain many new parameterized generalizations of the Rogers-Ramanujan type identities, such as a generalization of the analytic versions of the first and second Göllnitz-Gordon partition identities, and generalizations of the first, second, and third Rogers-Selberg identities. In Section 2, we mainly discuss the three-term recursions. In Section 3, we focus on four-term recursions. Moreover, in [6,12], the authors also found some new Rogers-Ramanujan type identities which are the partners to those in Slater's list. By using the determinant method, we can give the initial conditions of the recursions for these new identities, and then find the generalizations of these identities.

In [17], Sills gave an annotated and cross-referenced version of Slater's list of identities from [18] as an appendix. In this paper, we use this version of the list as the reference.

As usual, we follow the notation and terminology in [8]. For $|q| < 1$, the q-shifted factorial is defined by

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \quad \text{and} \quad (a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \text{ for } n \in \mathbb{C}.$$

For convenience, we shall adopt the following notation for multiple q-shifted factorials:

$$(a_1, a_2, \ldots, a_m; q)_n = (a_1;q)_n(a_2;q)_n \cdots (a_m;q)_n,$$

where n is an integer or infinity.

In order to sketch the paper clearly, we list the main results in a table.

Identities in Slater's list and some new ones	**Generalizations**
Identity A.8 (Gauss-Lebesgue [11]) Identity A.13 (Slater [18])	Theorem 2.1
Identity A.16 (Rogers [13]) Identity A.20 (Rogers [13])	Theorem 2.2
Identity A.29 (Slater [18]) Identity A.50 (Slater [18])	Theorem 2.3 (1) (2)
Identity A.34 (Slater [18]): The analytic version of the second Göllnitz-Gordon partition identity. Identity A.36 (Slater [18]): The analytic version of the first Göllnitz-Gordon partition identity.	Theorem 2.4
Identity A.38 (Slater [18]) Identity A.39 (Jackson [10])	Theorem 2.5 (1) (2)
Identity A.79 (Rogers [13]) Identity A.96 (Rogers [13])	Theorem 2.6 (1) (2)
Identity A.94 (Rogers [13]) Identity A.99 (Rogers [13])	Theorem 2.7 (1) (2)
Identity A.25 (Slater [18]) An identity (McLaughlin et al. [12, Eq. (2.7)])	Theorem 2.8
Identity A.31 (Rogers [14] and Selberg [16]) The third Rogers-Selberg identity Identity A.32 (Rogers [13] and Selberg [16]) The second Rogers-Selberg identity Identity A.33 (Rogers [13] and Selberg [16]) The first Rogers-Selberg identity	Theorem 3.1 (1) (2)
Identity A.59 (Rogers [14]) Identity A.60 (Rogers [14]) Identity A.61 (Rogers [13])	Theorem 3.2 (1) (2)
Identity A.80 (Rogers [14]) Identity A.81 (Rogers [14]) Identity A.82 (Rogers [14])	Theorem 3.3 (1) (2)
Identity A.117 (Slater [18]) Identity A.118 (Slater [18]) Identity A.119 (Slater [18])	Theorem 3.4 (1) (2)
Identity A.21 (Slater [18]) An identity (McLaughlin et al. [12, Eq. (2.5)]) An identity (Bowman et al. [6, Thm. 2.7])	Theorem 3.5 (1) (2)

2. Generalizations of identities with three-term recursions

In this section, we generalize the Rogers-Ramanujan type identities in Slater's list [18] by using the determinant method presented in [9]. Start with the three-term recursions of the polynomials which converge to the right hand sides of the identities in [17]. First, we construct a function $F(z)$ which is expressed by an infinite determinant. By expanding the determinant and comparing the coefficients,

we get a summation expression of $F(z)$. Then, we expand $D_n(z)$, a finite determinant of $F(z)$, to get a recursion which has appeared in Sills' list [17, Sec. 3.2]. Assume that the polynomials P_n and Q_n satisfy this recursion with different initial conditions, then $D_n(z)$ can be expressed by a linear combination of these two polynomials. By means of the initial conditions of $D_n(z)$, we get the limit of $D_n(z)$ which is another expression of $F(z)$. Finally, equating the two different expressions of $F(z)$, we obtain a new generalization.

In the following, for convenience, the recursions given by Sills [17] are directly presented below the identities in Slater's list.

Theorem 2.1. *We have*

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+2m-1)/2}}{(q;q)_n}=(-1)^m q^{-\binom{m}{2}}Q_{m-1}\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}-(-1)^m q^{-\binom{m}{2}}R_{m-1}\frac{(q^4;q^4)_\infty}{(q;q)_\infty}, \tag{2.1}$$

where

$$\begin{aligned} Q_m &= (1+q^{m-1})Q_{m-1}+q^{m-1}Q_{m-2}, \qquad Q_{-1}=1,\ Q_0=0,\ Q_1=1,\\ R_m &= (1+q^{m-1})R_{m-1}+q^{m-1}R_{m-2}, \qquad R_{-1}=-1,\ R_0=1,\ R_1=1. \end{aligned}$$

Proof. The identities A.8 and A.13 in Slater's list are stated as follows.

Identity A.8 (Gauss-Lebesgue [11]):

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_n}=\frac{(q^4;q^4)_\infty}{(q;q)_\infty}. \tag{2.2}$$

Sills [17] gave the following recursion for P_n which converge to the right hand side of (2.2).

$$P_n=(1+q^n)P_{n-1}+q^nP_{n-2}, \qquad P_{-1}=0,\ P_0=1,\ P_1=1+q. \tag{2.3}$$

Identity A.13 (Slater [18]):

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n-1)/2}}{(q;q)_n}=\frac{(q^4;q^4)_\infty}{(q;q)_\infty}+\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}, \tag{2.4}$$

$$P_n=(1+q^{n-1})P_{n-1}+q^{n-1}P_{n-2}, \qquad P_{-1}=0,\ P_0=1,\ P_1=2. \tag{2.5}$$

First, we need to shift the index n in (2.3) to let the two recursions coincide with each other. Letting $Q_n=P_{n-1}$ in (2.3), we get

$$Q_n=(1+q^{n-1})Q_{n-1}+q^{n-1}Q_{n-2}, \qquad Q_{-1}=1,\ Q_0=0,\ Q_1=1. \tag{2.6}$$

Thus, P_n in (2.5) and Q_n in (2.6) satisfy the same recursion with different initial conditions, and converge to the right hand sides of (2.4) and (2.2), respectively. In the following, we use P_n in (2.5) and Q_n in (2.6) to prove this theorem.

Then consider the following determinant:

$$F(z):=\begin{vmatrix} 1+z & zq & & & \cdots\\ -1 & 1+zq & zq^2 & & \cdots\\ & -1 & 1+zq^2 & zq^3 & \cdots\\ & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z)=(1+z)F(zq)+zqF(zq^2).$$

Setting

$$F(z)=\sum_{n=0}^{\infty} a_n z^n,$$

by comparing coefficients, we have

$$a_n = q^n a_n + q^{n-1}a_{n-1} + q^{2n-1}a_{n-1},$$
$$a_n = \frac{(1+q^n)q^{n-1}}{1-q^n}a_{n-1} = \cdots = \frac{(-q;q)_n q^{n(n-1)/2}}{(q;q)_n}a_0.$$

Since $a_0 = F(0) = 1$, iteration leads to

$$F(z)=\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n-1)/2}}{(q;q)_n}z^n,$$

and thus the left hand side of (2.1) can be expressed by $F(q^m)$.

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+z & zq & & & \cdots \\ -1 & 1+zq & zq^2 & & \cdots \\ & -1 & 1+zq^2 & zq^3 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1+zq^{n-2} & zq^{n-1} \\ & & & -1 & 1+zq^{n-1} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1+zq^{n-1})D_{n-1}(z) + zq^{n-1}D_{n-2}(z), \qquad D_0(z)=1,\ D_1(z)=1+z.$$

Then we have

$$D_{n-m}(q^m) = (1+q^{n-1})D_{n-m-1}(q^m) + q^{n-1}D_{n-m-2}(q^m). \tag{2.7}$$

According to (2.5), (2.6), and (2.7), we notice that the sequences $\langle D_{n-m}(q^m)\rangle_n$, $\langle P_n\rangle_n$, and $\langle Q_n\rangle_n$ satisfy the same recursion. Set

$$D_{n-m}(q^m) = \lambda_m P_n + \mu_m Q_n.$$

We can determine the parameters λ_m and μ_m using the initial conditions $D_0(q^m) = 1$, $D_1(q^m) = 1+q^m$, and the recursions (2.5) and (2.6), which leads to the evaluations

$$\lambda_m = \frac{Q_{m-1}}{P_m Q_{m-1} - P_{m-1}Q_m},$$
$$\mu_m = \frac{P_{m-1}}{P_{m-1}Q_m - P_m Q_{m-1}}.$$

Indeed, we have

$$P_m Q_{m-1} - P_{m-1}Q_m = (-1)^m q^{\binom{m}{2}},$$

which can be proved by induction on m.

Therefore, we have simpler forms for λ_m and μ_m as follows:

$$\lambda_m = (-1)^m q^{-\binom{m}{2}} Q_{m-1}, \qquad \mu_m = -(-1)^m q^{-\binom{m}{2}} P_{m-1}.$$

Notice that the above analysis has led to

$$D_{n-m}(q^m) = (-1)^m q^{-\binom{m}{2}} Q_{m-1}P_n - (-1)^m q^{-\binom{m}{2}} P_{m-1}Q_n.$$

Letting $n\to\infty$, we have

$$F(q^m) = (-1)^m q^{-\binom{m}{2}} Q_{m-1}P_\infty - (-1)^m q^{-\binom{m}{2}} P_{m-1}Q_\infty,$$

which is equivalent to the following identity

$$\sum_{n=0}^{\infty}\frac{(-q;q)_n q^{n(n+2m-1)/2}}{(q;q)_n} = (-1)^m q^{-\binom{m}{2}}\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}Q_{m-1} - (-1)^m q^{-\binom{m}{2}}\frac{(q^4;q^4)_\infty}{(q;q)_\infty}(P_{m-1}-Q_{m-1}).$$

Finally, set $R_{m-1}=P_{m-1}-Q_{m-1}$. According to (2.5) and (2.6), we have

$$R_m=(1+q^{m-1})R_{m-1}+q^{m-1}R_{m-2},\qquad R_{-1}=-1,\ R_0=1,\ R_1=1.$$

Therefore, we obtain (2.1) as desired. □

Setting $m=1$ and $m=0$ in (2.1), we get the identities (2.2) and (2.4), respectively.

Theorem 2.2. *We have*

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2mn}}{(q^4;q^4)_n}=\frac{A_m}{(q,q^4;q^5)_\infty(-q^2;q^2)_\infty}+\frac{B_m}{(q^2,q^3;q^5)_\infty(-q^2;q^2)_\infty},\tag{2.8}$$

where

$$\begin{aligned}A_m&=-q^{2m-3}A_{m-1}+A_{m-2}, &\qquad A_0=1,\ A_1=0,\\ B_m&=-q^{2m-3}B_{m-1}+B_{m-2}, &\qquad B_0=0,\ B_1=1.\end{aligned}$$

Proof. We state the identities A.16 and A.20 in Slater's list with the recursions given by Sills [17] as follows.

Identity A.16 (Rogers [13]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^4;q^4)_n}=\frac{1}{(q^2,q^3;q^5)_\infty(-q^2;q^2)_\infty},\tag{2.9}$$

$$P_n=(1-q^2+q^{2n+1})P_{n-1}+q^2P_{n-2},\qquad P_{-1}=1,\ P_0=1,\ P_1=1+q^3.\tag{2.10}$$

Identity A.20 (Rogers [13]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n}=\frac{1}{(q,q^4;q^5)_\infty(-q^2;q^2)_\infty},\tag{2.11}$$

$$P_n=(1-q^2+q^{2n-1})P_{n-1}+q^2P_{n-2},\qquad P_{-1}=1,\ P_0=1,\ P_1=1+q.\tag{2.12}$$

For the recursion (2.10), letting $Q_n=P_{n-1}$, we get

$$Q_n=(1-q^2+q^{2n-1})Q_{n-1}+q^2Q_{n-2},\qquad Q_{-1}=1-q^{-1},\ Q_0=1,\ Q_1=1.\tag{2.13}$$

Therefore, P_n in (2.12) and Q_n in (2.13) satisfy the same recursion with different initial conditions and converge to the right hand sides of (2.11) and (2.9), respectively.

Consider the following determinant:

$$F(z):=\begin{vmatrix}1-q^2+zq & q^2 & & & \cdots\\ -1 & 1-q^2+zq^3 & q^2 & & \cdots\\ & -1 & 1-q^2+zq^5 & q^2 & \cdots\\ & & \ddots & \ddots & \ddots\end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z)=(1-q^2+zq)F(zq^2)+q^2F(zq^4).$$

Setting

$$F(z)=\sum_{n=0}^{\infty}a_nz^n,$$

we obtain, upon comparing coefficients,

$$a_n=q^{2n}a_n-q^{2n+2}a_n+q^{2n-1}a_{n-1}+q^{4n+2}a_n,$$

$$a_n=\frac{q^{2n-1}}{(1-q^{2n})(1+q^{2n+2})}a_{n-1}=\cdots=\frac{q^{n^2}(1+q^2)}{(q^4;q^4)_n(1+q^{2n+2})}a_0.$$

In the following, we show some details for the calculation of a_0.

$F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1-q^2+zq & q^2 & & & \cdots \\ -1 & 1-q^2+zq^3 & q^2 & & \cdots \\ & -1 & 1-q^2+zq^5 & q^2 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1-q^2+zq^{2n-3} & q^2 \\ & & & -1 & 1-q^2+zq^{2n-1} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1-q^2+zq^{2n-1})D_{n-1}(z) + q^2 D_{n-2}(z), \qquad D_0(z)=1,\ D_1(z)=1-q^2+zq. \tag{2.14}$$

Since $a_0 = F(0) = \lim_{n\to\infty} D_n(0)$, according to the recursion (2.14), we have

$$D_n(0) = (1-q^2)D_{n-1}(0) + q^2 D_{n-2}(0), \qquad D_0(0)=1,\ D_1(0)=1-q^2.$$

Thus, we get the following recursion

$$\begin{aligned} D_n(0) - D_{n-1}(0) &= -q^2(D_{n-1}(0) - D_{n-2}(0)) \\ &= \cdots\cdots \\ &= (-1)^{n-1}q^{2n-2}(D_1(0) - D_0(0)) \\ &= (-1)^n q^{2n}. \end{aligned}$$

Then we have

$$D_n(0) = D_{n-1}(0) + (-1)^n q^{2n} = \cdots = \frac{1+(-1)^n q^{2n+2}}{1+q^2}.$$

Finally, letting $n\to\infty$ in $D_n(0)$, we get

$$a_0 = \lim_{n\to\infty} D_n(0) = \frac{1}{1+q^2}.$$

Therefore, we have

$$F(z) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n(1+q^{2n+2})} z^n,$$

and the left hand side of (2.8) can be expressed by $F(q^{2m}) + q^2F(q^{2m+2})$.

Due to (2.14), we have

$$D_{n-m}(q^{2m}) = (1-q^2+q^{2n-1})D_{n-m-1}(q^{2m}) + q^2 D_{n-m-2}(q^{2m}). \tag{2.15}$$

According to (2.12), (2.13), and (2.15), we notice that the sequences $\langle D_{n-m}(q^{2m})\rangle_n$, $\langle P_n\rangle_n$, and $\langle Q_n\rangle_n$ satisfy the same recursion. Set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n. \tag{2.16}$$

We can determine the parameters λ_m and μ_m using the initial conditions $D_0(q^{2m}) = 1$, $D_1(q^{2m}) = 1-q^2+q^{2m+1}$, and the recursions (2.12) and (2.13), which leads to the evaluations

$$\lambda_m = \frac{Q_{m-1}}{P_mQ_{m-1} - P_{m-1}Q_m},$$

$$\mu_m = \frac{P_{m-1}}{P_{m-1}Q_m - P_mQ_{m-1}}.$$

Indeed, we have

$$P_mQ_{m-1} - P_{m-1}Q_m = (-1)^{m-1}q^{2m-1},$$

which can be proved by induction on m. Then we have simpler forms for λ_m and μ_m as follows:

$$\lambda_m = (-1)^{m-1}q^{1-2m}Q_{m-1}, \qquad \mu_m = -(-1)^{m-1}q^{1-2m}P_{m-1}. \tag{2.17}$$

Now setting $m \to m+1$ in (2.16), we get

$$D_{n-m-1}(q^{2m+2}) = \lambda_{m+1} P_n + \mu_{m+1} Q_n.$$

Thus, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2+2mn}}{(q^4;q^4)_n} &= F(q^{2m}) + q^2 F(q^{2m+2}) \\ &= (\lambda_m + q^2\lambda_{m+1})P_\infty + (\mu_m + q^2\mu_{m+1})Q_\infty.\end{aligned}$$

According to (2.17), we get

$$\begin{aligned}\lambda_m + q^2\lambda_{m+1} &= (-1)^m q^{1-2m}(Q_m - Q_{m-1}), \\ \mu_m + q^2\mu_{m+1} &= (-1)^{m-1} q^{1-2m}(P_m - P_{m-1}).\end{aligned}$$

Setting $A_m = (-1)^m q^{1-2m}(Q_m - Q_{m-1})$, due to (2.13), we have

$$\begin{aligned}A_m &= xA_{m-1} + yA_{m-2} \\ &= x(-1)^{m-1}q^{3-2m}(Q_{m-1} - Q_{m-2}) + y(-1)^m q^{5-2m}(Q_{m-2} - Q_{m-3}) \\ &= [(-1)^{m-1}(1-q^{5-2m})x + (-1)^m q^{5-2m} y]Q_{m-2} + (-1)^{m-1}q^{5-2m}(x+y)Q_{m-3}.\end{aligned}$$

and

$$\begin{aligned}A_m &= (-1)^m q^{1-2m}(Q_m - Q_{m-1}) \\ &= (-1)^m (q^{5-2m} + q^{2m-3} - q^2)Q_{m-2} + (-1)^m (q^2 - q^{5-2m})Q_{m-3}.\end{aligned}$$

Therefore, we get

$$\begin{cases} (-1)^{m-1}(1-q^{5-2m})x + (-1)^m q^{5-2m} y = (-1)^m (q^{5-2m} + q^{2m-3} - q^2), \\ (-1)^{m-1} q^{5-2m}(x+y) = (-1)^m (q^2 - q^{5-2m}). \end{cases}$$

Then

$$\begin{cases} x = -q^{2m-3}, \\ y = 1, \end{cases}$$

which means that

$$A_m = -q^{2m-3}A_{m-1} + A_{m-2}, \qquad A_0 = 1,\ A_1 = 0.$$

Similarly, setting $B_m = (-1)^{m-1}q^{1-2m}(P_m - P_{m-1})$, we obtain

$$B_m = -q^{2m-3}B_{m-1} + B_{m-2}, \qquad B_0 = 0,\ B_1 = 1.$$

Thus the above analysis has led to (2.8). □

Setting $m = 1$ and $m = 0$ in (2.8), we get the identities (2.9) and (2.11), respectively.

Theorem 2.3. *We have*

(1)

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q;q)_{2n+1}} &= \frac{q^{1-m}(q^2,q^{10},q^{12};q^{12})_\infty}{(q;q^2)_{m-1}(q;q)_\infty} P_{m-1} \\ &\quad - \frac{q^{1-m}(-q^2,-q^4,q^6;q^6)_\infty(-q;q^2)_\infty}{(q;q^2)_{m-1}(q^2;q^2)_\infty} Q_{m-1}, \qquad (2.18)\end{aligned}$$

where

$$\begin{aligned}P_m &= (1+q+q^{2m-1})P_{m-1} + (q^{2m-2} - q)P_{m-2}, \qquad P_0 = 1,\ P_1 = 1+q, \\ Q_m &= (1+q+q^{2m-1})Q_{m-1} + (q^{2m-2} - q)Q_{m-2}, \qquad Q_0 = 0,\ Q_1 = 1.\end{aligned}$$

(2)

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q;q)_{2n}} = \frac{(-q^2,-q^4,q^6;q^6)_\infty(-q;q^2)_\infty}{(q;q^2)_m(q^2;q^2)_\infty} A_m - \frac{(q^2,q^{10},q^{12};q^{12})_\infty}{(q;q^2)_m(q;q)_\infty} B_m, \tag{2.19}$$

where

$$\begin{aligned} A_m &= (1+q+q^{2m-2})A_{m-1} + (q^{2m-2}-q)A_{m-2}, \qquad A_0 = 1,\ A_1 = 1, \\ B_m &= (1+q+q^{2m-2})B_{m-1} + (q^{2m-2}-q)B_{m-2}, \qquad B_0 = 0,\ B_1 = 2q. \end{aligned}$$

Proof. The identities A.29 and A.50 in Slater's list are stated as follows.

Identity A.29 (Slater [18]):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q)_{2n}} = \frac{(-q^2,-q^4,q^6;q^6)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{2.20}$$

$$P_n = (1+q+q^{2n-1})P_{n-1} + (q^{2n-2}-q)P_{n-2}, \qquad P_{-1} = -\frac{q}{1-q},\ P_0 = 1,\ P_1 = 1+q. \tag{2.21}$$

Identity A.50 (Slater [18]):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q;q)_{2n+1}} = \frac{(q^2,q^{10},q^{12};q^{12})_\infty}{(q;q)_\infty}, \tag{2.22}$$

$$P_n = (1+q+q^{2n+1})P_{n-1} + (q^{2n}-q)P_{n-2}, \qquad P_{-1} = 0,\ P_0 = 1,\ P_1 = 1+q+q^3. \tag{2.23}$$

For the recursion (2.23), letting $Q_n = P_{n-1}$, we get the recursion

$$Q_n = (1+q+q^{2n-1})Q_{n-1} + (q^{2n-2}-q)Q_{n-2}, \qquad Q_{-1} = \frac{1}{1-q},\ Q_0 = 0,\ Q_1 = 1. \tag{2.24}$$

The polynomials P_n in (2.21) and Q_n in (2.24) satisfy the same recursion with different initial conditions, and converge to the right hand sides of (2.20) and (2.22), respectively.

Consider the following determinant:

$$F(z) := \begin{vmatrix} 1+q+zq & zq^2-q & & & \cdots \\ -1 & 1+q+zq^3 & zq^4-q & & \cdots \\ & -1 & 1+q+zq^5 & zq^6-q & \cdots \\ & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z) = (1+q+zq)F(zq^2) + (zq^2-q)F(zq^4).$$

Setting

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we get, upon comparing coefficients,

$$\begin{aligned} a_n &= q^{2n}a_n + q^{2n+1}a_n + q^{2n-1}a_{n-1} + q^{4n-2}a_{n-1} - q^{4n+1}a_n, \\ a_n &= \frac{(1+q^{2n-1})q^{2n-1}}{(1-q^{2n})(1-q^{2n+1})} a_{n-1} = \cdots = \frac{(-q;q^2)_n q^{n^2}(1-q)}{(q;q)_{2n+1}} a_0. \end{aligned}$$

Resorting to the same technique for a_0 in the proof of Theorem 2.2, we have $a_0 = \frac{1}{1-q}$. Thus, we have

$$F(z) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q)_{2n+1}} z^n.$$

We observe that the left hand sides of (2.18) and (2.19) can be expressed by $F(q^{2m})$ and $F(q^{2m}) - qF(q^{2m+2})$, respectively.

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+q+zq & zq^2-q & & & \cdots \\ -1 & 1+q+zq^3 & zq^4-q & & \cdots \\ & -1 & 1+q+zq^5 & zq^6-q & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1+q+zq^{2n-3} & zq^{2n-2}-q \\ & & & -1 & 1+q+zq^{2n-1} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1+q+zq^{2n-1})D_{n-1}(z) + (zq^{2n-2}-q)D_{n-2}(z),$$
$$D_0(z) = 1,\ D_1(z) = 1+q+zq.$$

Then we have

$$D_{n-m}(q^{2m}) = (1+q+q^{2n-1})D_{n-m-1}(q^{2m}) + (q^{2n-2}-q)D_{n-m-2}(q^{2m}). \tag{2.25}$$

According to (2.21), (2.24), and (2.25), we notice that the sequences $\langle D_{n-m}(q^{2m})\rangle_n$, $\langle P_n\rangle_n$, and $\langle Q_n\rangle_n$ satisfy the same recursion. Set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n.$$

We can determine the parameters λ_m and μ_m using the initial conditions $D_0(q^{2m}) = 1$, $D_1(q^{2m}) = 1+q+q^{2m+1}$, and the recursions (2.21) and (2.24), which leads to the evaluations

$$\lambda_m = \frac{Q_{m-1}}{P_mQ_{m-1} - P_{m-1}Q_m},$$

$$\mu_m = \frac{P_{m-1}}{P_{m-1}Q_m - P_mQ_{m-1}}.$$

Notice that

$$P_mQ_{m-1} - P_{m-1}Q_m = -q^{m-1}(q;q^2)_{m-1},$$

which can be proved by induction on m. Then we have simpler forms for λ_m and μ_m as follows:

$$\lambda_m = -\frac{q^{1-m}}{(q;q^2)_{m-1}}Q_{m-1}, \qquad \mu_m = \frac{q^{1-m}}{(q;q^2)_{m-1}}P_{m-1}. \tag{2.26}$$

Therefore, we obtain Equation (2.18).

Meanwhile, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q;q)_{2n}} &= F(q^{2m}) - qF(q^{2m+2}) \\ &= (\lambda_m - q\lambda_{m+1})P_\infty + (\mu_m - q\mu_{m+1})Q_\infty.\end{aligned}$$

According to (2.26), we get

$$\lambda_m - q\lambda_{m+1} = \frac{q^{1-m}}{(q;q^2)_m}[Q_m - (1-q^{2m-1})Q_{m-1}],$$

$$\mu_m - q\mu_{m+1} = -\frac{q^{1-m}}{(q;q^2)_m}[P_m - (1-q^{2m-1})P_{m-1}].$$

Setting $A_m = q^{1-m}[Q_m - (1-q^{2m-1})Q_{m-1}]$ and $B_m = q^{1-m}[P_m - (1-q^{2m-1})P_{m-1}]$, we get Equation (2.19) as desired. $\square$

The identities (2.20) and (2.22) are the special cases of (2.19) and (2.18), respectively.

Theorem 2.4. *We have*

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q^2;q^2)_n} = \frac{(-1)^m q^{m-m^2} Q_{m-1}}{(q,q^4,q^7;q^8)_\infty} - \frac{(-1)^m q^{m-m^2} P_{m-1}}{(q^3,q^4,q^5;q^8)_\infty}, \tag{2.27}$$

where

$$P_m = (1+q^{2m-1})P_{m-1} + q^{2m-2}P_{m-2}, \qquad P_{-1}=0,\ P_0=1,\ P_1=1+q,$$
$$Q_m = (1+q^{2m-1})Q_{m-1} + q^{2m-2}Q_{m-2}, \qquad Q_{-1}=1,\ Q_0=0,\ Q_1=1.$$

Proof. We use the identities A.34 and A.36 in Slater's list to prove the theorem.

Identity A.34 (Slater [18]): The analytic version of the second Göllnitz-Gordon partition identity.[*]

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{1}{(q^3,q^4,q^5;q^8)_\infty}, \tag{2.28}$$

$$P_n = (1+q^{2n+1})P_{n-1} + q^{2n}P_{n-2}, \qquad P_{-1}=0,\ P_0=1,\ P_1=1+q^3. \tag{2.29}$$

Identity A.36 (Slater [18]): The analytic version of the first Göllnitz-Gordon partition identity.

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{1}{(q,q^4,q^7;q^8)_\infty}, \tag{2.30}$$

$$P_n = (1+q^{2n-1})P_{n-1} + q^{2n-2}P_{n-2}, \qquad P_{-1}=0,\ P_0=1,\ P_1=1+q. \tag{2.31}$$

For the recursion (2.29), letting $Q_n = P_{n-1}$, we get the recursion

$$Q_n = (1+q^{2n-1})Q_{n-1} + q^{2n-2}Q_{n-2}, \qquad Q_{-1}=1,\ Q_0=0,\ Q_1=1. \tag{2.32}$$

Therefore, P_n in (2.31) and Q_n in (2.32) converge to the right hand sides of (2.30) and (2.28), respectively. In the following, they are used to prove this theorem.

Consider the following determinant:

$$F(z) := \begin{vmatrix} 1+zq & zq^2 & & & \cdots \\ -1 & 1+zq^3 & zq^4 & & \cdots \\ & -1 & 1+zq^5 & zq^6 & \cdots \\ & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z) = (1+zq)F(zq^2) + zq^2F(zq^4).$$

Setting

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we get, upon comparing coefficients,

$$a_n = q^{2n}a_n + q^{2n-1}a_{n-1} + q^{4n-2}a_{n-1},$$

$$a_n = \frac{(1+q^{2n-1})q^{2n-1}}{1-q^{2n}} a_{n-1} = \cdots = \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} a_0.$$

[*]There is a typo in the recursion of Identity A.34 given by Sills [17]. In [4], Andrews et al. pointed out this recursion by considering a special case of the Al-Salam/Ismail polynomials [1].

Since $a_0 = 1$, iteration leads to

$$F(z) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n} z^n,$$

and thus the left hand side of (2.27) can be expressed by $F(q^{2m})$.

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+zq & zq^2 & & & \cdots \\ -1 & 1+zq^3 & zq^4 & & \cdots \\ & -1 & 1+zq^5 & zq^6 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1+zq^{2n-3} & zq^{2n-2} \\ & & & -1 & 1+zq^{2n-1} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1+zq^{2n-1})D_{n-1}(z) + zq^{2n-2}D_{n-2}(z), \qquad D_0(z) = 1,\ D_1(z) = 1+zq.$$

Then we have

$$D_{n-m}(q^{2m}) = (1+q^{2n-1})D_{n-m-1}(q^{2m}) + q^{2n-2}D_{n-m-2}(q^{2m}).$$

Therefore, we set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n.$$

Using the initial conditions $D_0(q^{2m}) = 1$ and $D_1(q^{2m}) = 1 + q^{2m+1}$, we get

$$\lambda_m = \frac{Q_{m-1}}{P_m Q_{m-1} - P_{m-1} Q_m},$$

$$\mu_m = \frac{P_{m-1}}{P_{m-1} Q_m - P_m Q_{m-1}}.$$

Indeed, we have

$$P_m Q_{m-1} - P_{m-1} Q_m = (-1)^m q^{m^2-m},$$

which can be proved by induction on m. Then we have simpler forms for λ_m and μ_m as follows:

$$\lambda_m = (-1)^m q^{m-m^2} Q_{m-1}, \qquad \mu_m = -(-1)^m q^{m-m^2} P_{m-1}.$$

Equation (2.27) is proved. □

The identities (2.28) and (2.30) are the special cases of Equation (2.27).

Theorem 2.5. *We have*

(1)

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q;q)_{2n+1}} = \frac{q^{1-m}(q^3,q^5,q^8;q^8)_\infty (q^2,q^{14};q^{16})_\infty}{(q;q^2)_{m-1}(q;q)_\infty} P_{m-1} - \frac{q^{1-m}(q,q^7,q^8;q^8)_\infty (q^6,q^{10};q^{16})_\infty}{(q;q^2)_{m-1}(q;q)_\infty} Q_{m-1}, \tag{2.33}$$

where

$$P_m = (1+q)P_{m-1} + (q^{2m-2} - q)P_{m-2}, \qquad P_0 = 1,\ P_1 = 1,$$
$$Q_m = (1+q)Q_{m-1} + (q^{2m-2} - q)Q_{m-2}, \qquad Q_0 = 0,\ Q_1 = 1.$$

(2)

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q;q)_{2n}} = \frac{(q,q^7,q^8;q^8)_\infty (q^6,q^{10};q^{16})_\infty}{(q;q^2)_m (q;q)_\infty} A_m - \frac{(q^3,q^5,q^8;q^8)_\infty (q^2,q^{14};q^{16})_\infty}{(q;q^2)_m (q;q)_\infty} B_m, \tag{2.34}$$

where

$$A_m = (1+q)A_{m-1} + (q^{2m-2} - q)A_{m-2}, \qquad A_0 = 1,\ A_1 = 1,$$

$$B_m = (1+q)B_{m-1} + (q^{2m-2} - q)B_{m-2}, \qquad B_0 = 0,\ B_1 = q.$$

Proof. We use the following identities A.38 and A.39 in Slater's list to prove the theorem.

Identity A.38 (Slater [18]):

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q)_{2n+1}} = \frac{(q^3,q^5,q^8;q^8)_\infty (q^2,q^{14};q^{16})_\infty}{(q;q)_\infty}, \tag{2.35}$$

$$P_n = (1+q)P_{n-1} + (q^{2n} - q)P_{n-2}, \qquad P_{-1} = 0,\ P_0 = 1,\ P_1 = 1+q. \tag{2.36}$$

Identity A.39 (Jackson [10]):

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n}} = \frac{(q,q^7,q^8;q^8)_\infty (q^6,q^{10};q^{16})_\infty}{(q;q)_\infty}, \tag{2.37}$$

$$P_n = (1+q)P_{n-1} + (q^{2n-2} - q)P_{n-2}, \qquad P_{-1} = -\frac{q}{1-q},\ P_0 = 1,\ P_1 = 1. \tag{2.38}$$

For the recursion (2.36), letting $Q_n = P_{n-1}$, we get the recursion

$$Q_n = (1+q)Q_{n-1} + (q^{2n-2} - q)Q_{n-2}, \qquad Q_{-1} = \frac{1}{1-q},\ Q_0 = 0,\ Q_1 = 1. \tag{2.39}$$

Therefore, P_n in (2.38) and Q_n in (2.39) satisfy the same recursion with different initial conditions, and converge to the right hand sides of (2.37) and (2.35), respectively.

Consider the following determinant:

$$F(z) := \begin{vmatrix} 1+q & zq^2-q & & & \cdots \\ -1 & 1+q & zq^4-q & & \cdots \\ & -1 & 1+q & zq^6-q & \cdots \\ & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z) = (1+q)F(zq^2) + (zq^2 - q)F(zq^4).$$

Setting

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we get, upon comparing coefficients,

$$a_n = q^{2n}a_n + q^{2n+1}a_n + q^{4n-2}a_{n-1} - q^{4n+1}a_n,$$

$$a_n = \frac{q^{4n-2}}{(1-q^{2n})(1-q^{2n+1})}a_{n-1} = \cdots = \frac{q^{2n^2}(1-q)}{(q;q)_{2n+1}}a_0.$$

Since $a_0 = \frac{1}{1-q}$, iteration leads to

$$F(z) = \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q;q)_{2n+1}} z^n,$$

and thus the left hand sides of (2.33) and (2.34) can be expressed by $F(q^{2m})$ and $F(q^{2m}) - qF(q^{2m+2})$, respectively.

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+q & zq^2-q & & & \cdots \\ -1 & 1+q & zq^4-q & & \cdots \\ & -1 & 1+q & zq^6-q & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1+q & zq^{2n-2}-q \\ & & & -1 & 1+q \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1+q)D_{n-1}(z) + (zq^{2n-2}-q)D_{n-2}(z), \qquad D_0(z)=1,\ D_1(z)=1+q.$$

Then we have

$$D_{n-m}(q^{2m}) = (1+q)D_{n-m-1}(q^{2m}) + (q^{2n-2}-q)D_{n-m-2}(q^{2m}).$$

Therefore, we set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n.$$

We can determine the parameters λ_m and μ_m using the initial conditions $D_0(q^{2m}) = 1$, $D_1(q^{2m}) = 1+q$, and the recursions (2.38) and (2.39), which leads to the evaluations

$$\lambda_m = \frac{Q_{m-1}}{P_m Q_{m-1} - P_{m-1} Q_m},$$

$$\mu_m = \frac{P_{m-1}}{P_{m-1} Q_m - P_m Q_{m-1}}.$$

We get

$$P_m Q_{m-1} - P_{m-1} Q_m = -q^{m-1}(q;q^2)_{m-1},$$

which can be proved by induction on m. Then we have

$$\lambda_m = -\frac{q^{1-m}}{(q;q^2)_{m-1}} Q_{m-1}, \qquad \mu_m = \frac{q^{1-m}}{(q;q^2)_{m-1}} P_{m-1}. \tag{2.40}$$

Therefore, Equation (2.33) is proved.

Furthermore, we have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q;q)_{2n}} &= F(q^{2m}) - qF(q^{2m+2}) \\ &= (\lambda_m - q\lambda_{m+1})P_\infty + (\mu_m - q\mu_{m+1})Q_\infty.\end{aligned}$$

According to (2.40), we get

$$\begin{aligned}\lambda_m - q\lambda_{m+1} &= \frac{q^{1-m}}{(q;q^2)_m}[Q_m - (1-q^{2m-1})Q_{m-1}], \\ \mu_m - q\mu_{m+1} &= -\frac{q^{1-m}}{(q;q^2)_m}[P_m - (1-q^{2m-1})P_{m-1}].\end{aligned}$$

Setting $A_m = q^{1-m}[Q_m - (1-q^{2m-1})Q_{m-1}]$ and $B_m = q^{1-m}[P_m - (1-q^{2m-1})P_{m-1}]$, we obtain Equation (2.34). ☐

The identities (2.35) and (2.37) are the special cases of (2.33) and (2.34), respectively.

Theorem 2.6. *We have*

(1)

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2mn}}{(q;q)_{2n+1}} = \frac{q^{1-m}(q^4,q^6,q^{10};q^{10})_\infty (q^2,q^{18};q^{20})_\infty}{(q;q)_\infty} P_{m-1}$$

$$-\frac{q^{1-m}(q^8,q^{12},q^{20};q^{20})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}Q_{m-1}, \tag{2.41}$$

where

$$P_m=(1+q+q^{2m-1})P_{m-1}-qP_{m-2}, \qquad P_{-1}=1,\ P_0=1,\ P_1=1+q,$$
$$Q_m=(1+q+q^{2m-1})Q_{m-1}-qQ_{m-2}, \qquad Q_{-1}=-q^{-1},\ Q_0=0,\ Q_1=1.$$

(2)

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2mn}}{(q;q)_{2n}}=\frac{(q^8,q^{12},q^{20};q^{20})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}A_m-\frac{(q^4,q^6,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty}B_m, \tag{2.42}$$

where

$$A_m=(1+q+q^{2m-2})A_{m-1}-qA_{m-2}, \qquad A_0=1,\ A_1=1,$$
$$B_m=(1+q+q^{2m-2})B_{m-1}-qB_{m-2}, \qquad B_0=0,\ B_1=q.$$

Proof. The identities A.79 and A.96 are stated as follows.

Identity A.79 (Rogers [13]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_{2n}}=\frac{(q^8,q^{12},q^{20};q^{20})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{2.43}$$

$$P_n=(1+q+q^{2n-1})P_{n-1}-qP_{n-2}, \qquad P_{-1}=1,\ P_0=1,\ P_1=1+q. \tag{2.44}$$

Identity A.96 (Rogers [13]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q)_{2n+1}}=\frac{(q^4,q^6,q^{10};q^{10})_\infty(q^2,q^{18};q^{20})_\infty}{(q;q)_\infty}, \tag{2.45}$$

$$P_n=(1+q+q^{2n+1})P_{n-1}-qP_{n-2}, \qquad P_{-1}=0,\ P_0=1,\ P_1=1+q+q^3. \tag{2.46}$$

For the recursion (2.46), letting $Q_n=P_{n-1}$, we get the recursion

$$Q_n=(1+q+q^{2n-1})Q_{n-1}-qQ_{n-2}, \qquad Q_{-1}=-q^{-1},\ Q_0=0,\ Q_1=1. \tag{2.47}$$

The polynomials P_n in (2.44) and Q_n in (2.47) satisfy the same recursion with different initial conditions, and converge to the right hand sides of (2.43) and (2.45), respectively.

Consider the following determinant:

$$F(z):=\begin{vmatrix} 1+q+zq & -q & & & \cdots \\ -1 & 1+q+zq^3 & -q & & \cdots \\ & -1 & 1+q+zq^5 & -q & \cdots \\ & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z)=(1+q+zq)F(zq^2)-qF(zq^4).$$

Setting

$$F(z)=\sum_{n=0}^{\infty}a_nz^n,$$

we get, upon comparing coefficients,

$$a_n=q^{2n}a_n+q^{2n+1}a_n+q^{2n-1}a_{n-1}-q^{4n+1}a_n,$$

$$a_n=\frac{q^{2n-1}}{(1-q^{2n})(1-q^{2n+1})}a_{n-1}=\cdots=\frac{q^{n^2}(1-q)}{(q;q)_{2n+1}}a_0.$$

Since $a_0 = \frac{1}{1-q}$, we have

$$F(z) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_{2n+1}} z^n,$$

and thus the left hand sides of (2.41) and (2.42) can be expressed by $F(q^{2m})$ and $F(q^{2m})-qF(q^{2m+2})$, respectively.

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+q+zq & -q & & & \cdots \\ -1 & 1+q+zq^3 & -q & & \cdots \\ & -1 & 1+q+zq^5 & -q & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1+q+zq^{2n-3} & -q \\ & & & -1 & 1+q+zq^{2n-1} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1+q+zq^{2n-1})D_{n-1}(z) - qD_{n-2}(z), \qquad D_0(z)=1,\ D_1(z)=1+q+zq.$$

Then we have

$$D_{n-m}(q^{2m}) = (1+q+q^{2n-1})D_{n-m-1}(q^{2m}) - qD_{n-m-2}(q^{2m}).$$

Therefore, we set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n.$$

According to the initial conditions $D_0(q^{2m}) = 1$ and $D_1(q^{2m}) = 1+q+q^{2m+1}$, we have

$$\lambda_m = \frac{Q_{m-1}}{P_m Q_{m-1} - P_{m-1} Q_m},$$

$$\mu_m = \frac{P_{m-1}}{P_{m-1} Q_m - P_m Q_{m-1}}.$$

Indeed, we have

$$P_m Q_{m-1} - P_{m-1} Q_m = -q^{m-1},$$

which can be proved by induction on m. Then we have

$$\lambda_m = -q^{1-m} Q_{m-1}, \qquad \mu_m = q^{1-m} P_{m-1}. \tag{2.48}$$

Therefore, we obtain Equation (2.41).

Furthermore, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2+2mn}}{(q;q)_{2n}} &= F(q^{2m}) - qF(q^{2m+2}) \\ &= (\lambda_m - q\lambda_{m+1})P_\infty + (\mu_m - q\mu_{m+1})Q_\infty. \end{aligned}$$

According to (2.48), we get

$$\lambda_m - q\lambda_{m+1} = q^{1-m}(Q_m - Q_{m-1}),$$

$$\mu_m - q\mu_{m+1} = -q^{1-m}(P_m - P_{m-1}).$$

Setting $A_m = q^{1-m}(Q_m - Q_{m-1})$ and $B_m = q^{1-m}(P_m - P_{m-1})$, we obtain Equation (2.42). $\square$

The identities (2.43) and (2.45) are the special cases of (2.42) and (2.41), respectively.

Theorem 2.7. *We have*

(1)

$$\sum_{n=0}^{\infty}\frac{q^{n^2+(2m+1)n}}{(q;q)_{2n+1}}=\frac{q^{-m}(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty}Q_{m-1} - \frac{q^{-m}(q,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty}P_{m-1}, \tag{2.49}$$

where

$$P_m=(1+q+q^{2m})P_{m-1}-qP_{m-2},\qquad P_{-1}=0,\ P_0=1,\ P_1=1+q+q^2,$$
$$Q_m=(1+q+q^{2m})Q_{m-1}-qQ_{m-2},\qquad Q_{-1}=1,\ Q_0=1,\ Q_1=1+q^2.$$

(2)

$$\sum_{n=0}^{\infty}\frac{q^{n^2+(2m+1)n}}{(q;q)_{2n}}=\frac{(q,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty}A_m-\frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty}B_m, \tag{2.50}$$

where

$$A_m=(1+q+q^{2m-1})A_{m-1}-qA_{m-2},\qquad A_0=1,\ A_1=1+q,$$
$$B_m=(1+q+q^{2m-1})B_{m-1}-qB_{m-2},\qquad B_0=0,\ B_1=q.$$

Proof. We state the identities A.94 and A.99 in Slater's list as follows.

Identity A.94 (Rogers [13]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_{2n+1}}=\frac{(q^3,q^7,q^{10};q^{10})_\infty(q^4,q^{16};q^{20})_\infty}{(q;q)_\infty}, \tag{2.51}$$

$$P_n=(1+q+q^{2n})P_{n-1}-qP_{n-2},\qquad P_{-1}=0,\ P_0=1,\ P_1=1+q+q^2. \tag{2.52}$$

Identity A.99 (Rogers [13]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_{2n}}=\frac{(q,q^9,q^{10};q^{10})_\infty(q^8,q^{12};q^{20})_\infty}{(q;q)_\infty}, \tag{2.53}$$

$$Q_n=(1+q+q^{2n})Q_{n-1}-qQ_{n-2},\qquad Q_{-1}=1,\ Q_0=1,\ Q_1=1+q^2. \tag{2.54}$$

Consider the following determinant:

$$F(z):=\begin{vmatrix} 1+q+zq^2 & -q & & & \cdots \\ -1 & 1+q+zq^4 & -q & & \cdots \\ & -1 & 1+q+zq^6 & -q & \cdots \\ & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z)=(1+q+zq^2)F(zq^2)-qF(zq^4).$$

Setting

$$F(z)=\sum_{n=0}^{\infty}a_nz^n,$$

we get, upon comparing coefficients,

$$a_n=q^{2n}a_n+q^{2n+1}a_n+q^{2n}a_{n-1}-q^{4n+1}a_n,$$

$$a_n=\frac{q^{2n}}{(1-q^{2n})(1-q^{2n+1})}a_{n-1}=\cdots=\frac{q^{n^2+n}(1-q)}{(q;q)_{2n+1}}a_0.$$

Since $a_0 = \frac{1}{1-q}$, we have

$$F(z) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_{2n+1}} z^n,$$

and thus the left hand sides of (2.49) and (2.50) can be expressed by $F(q^{2m})$ and $F(q^{2m}) - qF(q^{2m+2})$, respectively.

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+q+zq^2 & -q & & & \cdots \\ -1 & 1+q+zq^4 & -q & & \cdots \\ & -1 & 1+q+zq^6 & -q & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1+q+zq^{2n-2} & -q \\ & & & -1 & 1+q+zq^{2n} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1+q+zq^{2n})D_{n-1}(z) - qD_{n-2}(z), \qquad D_0(z) = 1,\ D_1(z) = 1+q+zq^2.$$

Then we have

$$D_{n-m}(q^{2m}) = (1+q+q^{2n})D_{n-m-1}(q^{2m}) - qD_{n-m-2}(q^{2m}).$$

Set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n.$$

Using the initial conditions $D_0(q^{2m}) = 1$ and $D_1(q^{2m}) = 1+q+q^{2m+2}$, we get

$$\lambda_m = \frac{Q_{m-1}}{P_m Q_{m-1} - P_{m-1} Q_m},$$
$$\mu_m = \frac{P_{m-1}}{P_{m-1} Q_m - P_m Q_{m-1}}.$$

Indeed, we have

$$P_m Q_{m-1} - P_{m-1} Q_m = q^m,$$

which can be proved by induction on m. Then we have simpler forms for λ_m and μ_m as follows:

$$\lambda_m = q^{-m} Q_{m-1}, \qquad \mu_m = -q^{-m} P_{m-1}. \tag{2.55}$$

Therefore, we obtain Equation (2.49).

Furthermore, we have

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{q^{n^2+(2m+1)n}}{(q;q)_{2n}} &= F(q^{2m}) - qF(q^{2m+2}) \\ &= (\lambda_m - q\lambda_{m+1})P_\infty + (\mu_m - q\mu_{m+1})Q_\infty. \end{aligned}$$

According to (2.55), we get

$$\lambda_m - q\lambda_{m+1} = -q^{-m}(Q_m - Q_{m-1}),$$
$$\mu_m - q\mu_{m+1} = q^{-m}(P_m - P_{m-1}).$$

Setting $A_m = q^{-m}(P_m - P_{m-1})$ and $B_m = q^{-m}(Q_m - Q_{m-1})$, we get Equation (2.50). □

The identities (2.51) and (2.53) are the special cases of (2.49) and (2.50), respectively.

Theorem 2.8. *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q^4;q^4)_n} &= \frac{(q^6;q^6)_\infty}{(-q^2;q^2)_{m-1}(q^4;q^4)_\infty(q^3,q^9;q^{12})_\infty} A_m \\ &\quad - \frac{(q^3,q^3,q^6;q^6)_\infty(-q;q^2)_\infty}{(-q^2;q^2)_{m-1}(q^2;q^2)_\infty} B_m, \end{aligned} \tag{2.56}$$

where

$$A_m = -q^{2m-3}A_{m-1} + (1+q^{2m-4})A_{m-2}, \qquad A_0 = 0,\ A_1 = 1,$$
$$B_m = -q^{2m-3}B_{m-1} + (1+q^{2m-4})B_{m-2}, \qquad B_0 = -\frac{1}{2},\ B_1 = 0.$$

Proof. The identity A.25 in Slater's list is stated as follows:

Identity A.25 (Slater [18]):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \frac{(q^3,q^3,q^6;q^6)_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{2.57}$$

Sills [17] gave the following recursion for (2.57).

$$P_n = (1-q^2+q^{2n-1})P_{n-1} + (q^2+q^{2n-2})P_{n-2}, \qquad P_{-1} = \frac{q^2}{1+q^2},\ P_0 = 1,\ P_1 = 1+q. \tag{2.58}$$

Recently, McLaughlin et al. [12] found a partner to Equation (2.57).

An identity (McLaughlin et al. [12, Eq. (2.7)]):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(q^6;q^6)_\infty}{(q^4;q^4)_\infty(q^3,q^9;q^{12})_\infty}. \tag{2.59}$$

For this identity, we also have

$$Q_n = (1-q^2+q^{2n-1})Q_{n-1} + (q^2+q^{2n-2})Q_{n-2}, \tag{2.60}$$

where P_n and Q_n converge to the right hand sides of (2.57) and (2.59), respecitively. The initial conditions for Q_n is given in the following analysis.

Consider the following determinant:

$$F(z) := \begin{vmatrix} 1-q^2+zq & q^2+zq^2 & & & \cdots \\ -1 & 1-q^2+zq^3 & q^2+zq^4 & & \cdots \\ & -1 & 1-q^2+zq^5 & q^2+zq^6 & \cdots \\ & & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z) = (1-q^2+zq)F(zq^2) + (q^2+zq^2)F(zq^4).$$

Setting

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we get, upon comparing coefficients,

$$a_n = q^{2n}a_n - q^{2n+2}a_n + q^{2n-1}a_{n-1} + q^{4n+2}a_n + q^{4n-2}a_{n-1},$$

$$a_n = \frac{(1+q^{2n-1})q^{2n-1}}{(1-q^{2n})(1+q^{2n+2})}a_{n-1} = \cdots = \frac{(-q;q^2)_n q^{n^2}(1+q^2)}{(q^4;q^4)_n(1+q^{2n+2})}a_0.$$

Since $a_0 = \frac{1}{1+q^2}$, iteration leads to

$$F(z) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n(1+q^{2n+2})} z^n,$$

and thus the left hand side of (2.56) can be expressed by $F(q^{2m}) + q^2F(q^{2m+2})$.

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1-q^2+zq & q^2+zq^2 & & & \cdots \\ -1 & 1-q^2+zq^3 & q^2+zq^4 & & \cdots \\ & -1 & 1-q^2+zq^5 & q^2+zq^6 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ & & -1 & 1-q^2+zq^{2n-3} & q^2+zq^{2n-2} \\ & & & -1 & 1-q^2+zq^{2n-1} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1-q^2+zq^{2n-1})D_{n-1}(z) + (q^2+zq^{2n-2})D_{n-2}(z),$$
$$D_0(z) = 1,\ D_1(z) = 1-q^2+zq.$$

Then we have

$$D_{n-m}(q^{2m}) = (1-q^2+q^{2n-1})D_{n-m-1}(q^{2m}) + (q^2+q^{2n-2})D_{n-m-2}(q^{2m}).$$

Noticing that Q_∞ is $F(q^2)+q^2F(q^4)$, we have

$$Q_n = D_{n-1}(q^2) + q^2 D_{n-2}(q^4).$$

then we get the initial conditions for Q_n: $Q_0 = 1/2$ and $Q_1 = 1$.

Since the sequences $\langle D_{n-m}(q^{2m})\rangle_n$, $\langle P_n\rangle_n$, and $\langle Q_n\rangle_n$ satisfy the same recursion, we set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n.$$

According to the initial conditions $D_0(q^{2m}) = 1$ and $D_1(q^{2m}) = 1-q^2+q^{2m+1}$, we have

$$\lambda_m = \frac{Q_{m-1}}{P_m Q_{m-1} - P_{m-1}Q_m},$$
$$\mu_m = \frac{P_{m-1}}{P_{m-1}Q_m - P_m Q_{m-1}}.$$

Indeed, we have

$$P_m Q_{m-1} - P_{m-1}Q_m = (-1)^m q^{2m-2}(1-q)(-q^2;q^2)_{m-2},$$

which can be proved by induction on m.

Therefore, we have simpler forms for λ_m and μ_m as follows:

$$\lambda_m = \frac{(-1)^m q^{2-2m}}{(1-q)(-q^2;q^2)_{m-2}} Q_{m-1}, \qquad \mu_m = -\frac{(-1)^m q^{2-2m}}{(1-q)(-q^2;q^2)_{m-2}} P_{m-1}. \tag{2.61}$$

Moreover, we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q^4;q^4)_n} &= F(q^{2m}) + q^2F(q^{2m+2}) \\ &= (\lambda_m + q^2\lambda_{m+1})P_\infty + (\mu_m + q^2\mu_{m+1})Q_\infty.\end{aligned}$$

According to (2.61), we get

$$\lambda_m + q^2\lambda_{m+1} = -\frac{(-1)^m q^{2-2m}}{(1-q)(-q^2;q^2)_{m-1}}[Q_m - (1+q^{2m-2})Q_{m-1}],$$
$$\mu_m + q^2\mu_{m+1} = \frac{(-1)^m q^{2-2m}}{(1-q)(-q^2;q^2)_{m-1}}[P_m - (1+q^{2m-2})P_{m-1}].$$

Setting

$$A_m = (-1)^m q^{2-2m}[P_m - (1+q^{2m-2})P_{m-1}]/(1-q),$$
$$B_m = (-1)^m q^{2-2m}[Q_m - (1+q^{2m-2})Q_{m-1}]/(1-q),$$

we get Equation (2.56). □

The identities (2.57) and (2.59) are the special cases of (2.56), respectively.

3. Generalizations of identities with four-term recursions

In this section, we apply the determinant method to the Rogers-Ramanujan type identities with the four-term recursions of the polynomials which converge to the right hand sides of the identities in [17]. Moreover, we generalize some new identities in recent papers [6, 12]. During the calculation, some properties of determinants are used to simplify the identities.

Three identities are used to prove each theorem. For convenience, we give the same recursions for the polynomials P_n, Q_n, and R_n by shifting the index of the recursions given by Sills in [17], like the way we have done in the previous section, where P_n, Q_n, and R_n converge to the right hand sides of the identities in Slater's list .

Theorem 3.1. *We have*

(1)

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2mn}}{(q^2;q^2)_n(-q;q)_{2n+1}}=\frac{q^{-m}(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}A_m+\frac{q^{-m}(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}B_m+\frac{q^{-m}(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}C_m, \tag{3.1}$$

where

$$A_m=-(1+q^{2m-4})A_{m-1}+q^2A_{m-2}+q^2A_{m-3},\qquad A_0=-q,\ A_1=q,\ A_2=-q,$$

$$B_m=-(1+q^{2m-4})B_{m-1}+q^2B_{m-2}+q^2B_{m-3},\qquad B_0=0,\ B_1=0,\ B_2=q,$$

$$C_m=-(1+q^{2m-4})C_{m-1}+q^2C_{m-2}+q^2C_{m-3},\qquad C_0=1,\ C_1=0,\ C_2=0.$$

(2)

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2mn}}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}E_m+\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}F_m+\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty}G_m, \tag{3.2}$$

where

$$E_m=-(q+q^{2m-3})E_{m-1}+E_{m-2}+qE_{m-3},\qquad E_0=0,\ E_1=0,\ E_2=q,$$

$$F_m=-(q+q^{2m-3})F_{m-1}+F_{m-2}+qF_{m-3},\qquad F_0=0,\ F_1=1,\ F_2=-q,$$

$$G_m=-(q+q^{2m-3})G_{m-1}+G_{m-2}+qG_{m-3},\qquad G_0=1,\ G_1=0,\ G_2=1.$$

Proof. The identities A.31, A.32, and A.33 in Slater's list are stated as follows.

Identity A.31 (Rogers [14] and Selberg [16]): The third Rogers-Selberg identity.

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^2;q^2)_n(-q;q)_{2n+1}}=\frac{(q,q^6,q^7;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{3.3}$$

$$P_n=(1-q-q^2)P_{n-1}+(q^{2n}-q^3+q^2+q)P_{n-2}+q^3P_{n-3},$$
$$P_0=1,\ P_1=1-q,\ P_2=1-q+q^2+q^4. \tag{3.4}$$

Identity A.32 (Rogers [13] and Selberg [16]): The second Rogers-Selberg identity.

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2n}}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{(q^2,q^5,q^7;q^7)_\infty}{(q^2;q^2)_\infty}, \tag{3.5}$$

$$Q_n=(1-q-q^2)Q_{n-1}+(q^{2n}-q^3+q^2+q)Q_{n-2}+q^3Q_{n-3},$$
$$Q_0=1,\ Q_1=1,\ Q_2=1+q^4. \tag{3.6}$$

Identity A.33 (Rogers [13] and Selberg [16]): The first Rogers-Selberg identity.

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n}}=\frac{(q^3,q^4,q^7;q^7)_\infty}{(q^2;q^2)_\infty},\tag{3.7}$$

$$\begin{aligned}R_n&=(1-q-q^2)R_{n-1}+(q^{2n}-q^3+q^2+q)R_{n-2}+q^3R_{n-3},\\ R_0&=1,\ R_1=1+q^2,\ R_2=1+q^2-q^3.\end{aligned}\tag{3.8}$$

The polynomials P_n, Q_n, and R_n converge to the right hand sides of (3.3), (3.5), and (3.7), respectively.

Consider the following determinant:

$$F(z):=\begin{vmatrix}1-q-q^2 & zq^2-q^3+q^2+q & q^3 & & & \cdots\\ -1 & 1-q-q^2 & zq^4-q^3+q^2+q & q^3 & & \cdots\\ & -1 & 1-q-q^2 & zq^6-q^3+q^2+q & q^3 & \cdots\\ & & \ddots & \ddots & \ddots & \ddots\end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z)=(1-q-q^2)F(zq^2)+(zq^2-q^3+q^2+q)F(zq^4)+q^3F(zq^6).$$

Setting

$$F(z)=\sum_{n=0}^{\infty}a_nz^n,$$

we get, upon comparing coefficients,

$$a_n=q^{2n}a_n-q^{2n+1}a_n-q^{2n+2}a_n+q^{4n-2}a_{n-1}-q^{4n+3}a_n+q^{4n+2}a_n+q^{4n+1}a_n+q^{6n+3}a_n,$$

$$a_n=\frac{q^{4n-2}}{(1-q^{2n})(1+q^{2n+1})(1+q^{2n+2})}a_{n-1}=\cdots=\frac{q^{2n^2}(1+q)(1+q^2)}{(q^2;q^2)_n(-q;q)_{2n+2}}a_0.$$

Since $a_0=\frac{1}{(1+q)(1+q^2)}$, we have

$$F(z)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{(q^2;q^2)_n(-q;q)_{2n+2}}z^n.$$

Thus we get

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2mn}}{(q^2;q^2)_n(-q;q)_{2n+1}}=F(q^{2m})+q^2F(q^{2m+2}),\tag{3.9}$$

$$\sum_{n=0}^{\infty}\frac{q^{2n^2+2mn}}{(q^2;q^2)_n(-q;q)_{2n}}=F(q^{2m})+(q+q^2)F(q^{2m+2})+q^3F(q^{2m+4}).\tag{3.10}$$

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z):=\begin{vmatrix}1-q-q^2 & zq^2-q^3+q^2+q & q^3 & & \cdots\\ -1 & 1-q-q^2 & zq^4-q^3+q^2+q & q^3 & \cdots\\ \vdots & \ddots & \ddots & \ddots & \ddots\\ & & -1 & 1-q-q^2 & zq^{2n-2}-q^3+q^2+q\\ & & & -1 & 1-q-q^2\end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$\begin{aligned}D_n(z)&=(1-q-q^2)D_{n-1}(z)+(zq^{2n-2}-q^3+q^2+q)D_{n-2}(z)+q^3D_{n-3}(z),\\ D_0(z)&=1,\ D_1(z)=1-q-q^2,\ D_2(z)=1-q+q^3+q^4+zq^2.\end{aligned}$$

Then we have

$$D_{n-m+1}(q^{2m})=(1-q-q^2)D_{n-m}(q^{2m})+(q^{2n}-q^3+q^2+q)D_{n-m-1}(q^{2m})+q^3D_{n-m-2}(q^{2m}).$$

Since $\langle D_{n-m+1}(q^{2m})\rangle_n$, $\langle P_n\rangle_n$, $\langle Q_n\rangle_n$, and $\langle R_n\rangle_n$ satisfy the same recursion, we set

$$D_{n-m+1}(q^{2m})=\lambda_mP_n+\mu_mQ_n+\nu_mR_n.$$

Using the initial conditions $D_0(q^{2m}) = 1$, $D_1(q^{2m}) = 1-q-q^2$, and $D_2(q^{2m}) = 1-q+q^3+q^4+q^{2m+2}$, we have

$$\lambda_m = \frac{\begin{vmatrix} 1 & Q_{m-1} & R_{m-1} \\ 1-q-q^2 & Q_m & R_m \\ 1-q+q^3+q^4+q^{2m+2} & Q_{m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\mu_m = \frac{\begin{vmatrix} P_{m-1} & 1 & R_{m-1} \\ P_m & 1-q-q^2 & R_m \\ P_{m+1} & 1-q+q^3+q^4+q^{2m+2} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\nu_m = \frac{\begin{vmatrix} P_{m-1} & Q_{m-1} & 1 \\ P_m & Q_m & 1-q-q^2 \\ P_{m+1} & Q_{m+1} & 1-q+q^3+q^4+q^{2m+2} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}}.$$

Indeed, we have

$$\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix} = -q^{3m+2}. \tag{3.11}$$

The proof of (3.11) is by induction on m. The case $m = 0$ is trivial.

$$\begin{vmatrix} P_0 & Q_0 & R_0 \\ P_1 & Q_1 & R_1 \\ P_2 & Q_2 & R_2 \end{vmatrix} = -q^5.$$

The recursions (3.4), (3.6), (3.8), and some properties of determinants are used in the following induction step.

$$\begin{aligned}
&\begin{vmatrix} P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \\ P_{m+2} & Q_{m+2} & R_{m+2} \end{vmatrix} \\
&= \begin{vmatrix} P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \\ (1-q-q^2)P_{m+1} & (1-q-q^2)Q_{m+1} & (1-q-q^2)R_{m+1} \end{vmatrix} \\
&\quad + \begin{vmatrix} P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \\ (q^{2m+4}-q^3+q^2+q)P_m & (q^{2m+4}-q^3+q^2+q)Q_m & (q^{2m+4}-q^3+q^2+q)R_m \end{vmatrix} \\
&\quad + \begin{vmatrix} P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \\ q^3P_{m-1} & q^3Q_{m-1} & q^3R_{m-1} \end{vmatrix} \\
&= q^3 \begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}.
\end{aligned}$$

Therefore, we have simpler forms for λ_m, μ_m, and ν_m as follows:

$$\lambda_m = -\frac{\begin{vmatrix} 1 & Q_{m-1} & R_{m-1} \\ 1-q-q^2 & Q_m & R_m \\ 1-q+q^3+q^4+q^{2m+2} & Q_{m+1} & R_{m+1} \end{vmatrix}}{q^{3m+2}}, \tag{3.12}$$

$$\mu_m = -\frac{\begin{vmatrix} P_{m-1} & 1 & R_{m-1} \\ P_m & 1-q-q^2 & R_m \\ P_{m+1} & 1-q+q^3+q^4+q^{2m+2} & R_{m+1} \end{vmatrix}}{q^{3m+2}},$$

$$\nu_m = -\frac{\begin{vmatrix} P_{m-1} & Q_{m-1} & 1 \\ P_m & Q_m & 1-q-q^2 \\ P_{m+1} & Q_{m+1} & 1-q+q^3+q^4+q^{2m+2} \end{vmatrix}}{q^{3m+2}}.$$

According to (3.9) and (3.10), by setting

$$\begin{cases} A_m = q^m(\lambda_m + q^2\lambda_{m+1}), \\ B_m = q^m(\mu_m + q^2\mu_{m+1}), \\ C_m = q^m(\nu_m + q^2\nu_{m+1}), \end{cases} \quad \text{and} \quad \begin{cases} E_m = \lambda_m + (q+q^2)\lambda_{m+1} + q^3\lambda_{m+2}, \\ F_m = \mu_m + (q+q^2)\mu_{m+1} + q^3\mu_{m+2}, \\ G_m = \nu_m + (q+q^2)\nu_{m+1} + q^3\nu_{m+2}, \end{cases}$$

we have

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q^2;q^2)_n(-q;q)_{2n+1}} = q^{-m}A_mP_\infty + q^{-m}B_mQ_\infty + q^{-m}C_mR_\infty,$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+2mn}}{(q^2;q^2)_n(-q;q)_{2n}} = E_mP_\infty + F_mQ_\infty + G_mR_\infty.$$

In the following, we only present the calculation for $A_m = q^m(\lambda_m + q^2\lambda_{m+1})$. Others are similar.

According to (3.12), using the same technique in the proof of (3.11), we have

$$\begin{aligned} A_m &= q^m(\lambda_m + q^2\lambda_{m+1}) \\ &= -\frac{\begin{vmatrix} 1 & Q_{m-1} & R_{m-1} \\ 1-q & Q_m & R_m \\ 1-q+q^2+q^{2m+2} & Q_{m+1} & R_{m+1} \end{vmatrix}}{q^{2m+2}} \\ &= -\frac{\begin{vmatrix} 0 & Q_{m-2} & R_{m-2} \\ 1 & Q_{m-1} & R_{m-1} \\ 1-q & Q_m & R_m \end{vmatrix}}{q^{2m-1}}. \end{aligned}$$

Then we calculate A_{m-1}, A_{m-2}, and A_{m-3}. Letting the last two columns in the determinants of A_{m-1}, A_{m-2}, and A_{m-3} be the same as those of A_m, we set $A_m = xA_{m-1} + yA_{m-2} + zA_{m-3}$. Solve the equation, we get

$$A_m = -(1+q^{2m-4})A_{m-1} + q^2A_{m-2} + q^2A_{m-3}.$$

Using (3.12) and the initial conditions of P_n, Q_n, and R_n, we have $A_0 = -q$, $A_1 = q$, and $A_2 = -q$. Following the same way, we calculate the recursions of B_m, C_m, E_m, F_m, and G_m in turn. Then we obtain (3.1) and (3.2). □

Notice that (3.3) is a special case of (3.1), and (3.5) and (3.7) are the special cases of (3.2).

Theorem 3.2. *We have*

(1)

$$\sum_{n=0}^{\infty}\frac{q^{n^2+mn}}{(q;q^2)_{n+1}(q;q)_n}=\frac{(q^2,q^{12},q^{14};q^{14})_\infty}{(q;q)_\infty}\lambda_m+\frac{(q^4,q^{10},q^{14};q^{14})_\infty}{(q;q)_\infty}\mu_m + \frac{(q^6,q^8,q^{14};q^{14})_\infty}{(q;q)_\infty}\nu_m, \tag{3.13}$$

where

$$\begin{aligned}
\lambda_m &= (1+q^{m-3})\lambda_{m-1}+q^{-1}\lambda_{m-2}-q^{-1}\lambda_{m-3}, \qquad \lambda_0=q,\ \lambda_1=0,\ \lambda_2=1,\\
\mu_m &= (1+q^{m-3})\mu_{m-1}+q^{-1}\mu_{m-2}-q^{-1}\mu_{m-3}, \qquad \mu_0=0,\ \mu_1=1,\ \mu_2=0,\\
\nu_m &= (1+q^{m-3})\nu_{m-1}+q^{-1}\nu_{m-2}-q^{-1}\nu_{m-3}, \qquad \nu_0=1,\ \nu_1=0,\ \nu_2=0.
\end{aligned}$$

(2)

$$\sum_{n=0}^{\infty}\frac{q^{n^2+mn}}{(q;q^2)_{n}(q;q)_n}=\frac{(q^2,q^{12},q^{14};q^{14})_\infty}{(q;q)_\infty}E_m+\frac{(q^4,q^{10},q^{14};q^{14})_\infty}{(q;q)_\infty}F_m+\frac{(q^6,q^8,q^{14};q^{14})_\infty}{(q;q)_\infty}G_m, \tag{3.14}$$

where

$$\begin{aligned}
E_m &= (1+q^{m-1})E_{m-1}+qE_{m-2}-qE_{m-3}, \qquad E_0=0,\ E_1=-q,\ E_2=-q-q^2,\\
F_m &= (1+q^{m-1})F_{m-1}+qF_{m-2}-qF_{m-3}, \qquad F_0=0,\ F_1=0,\ F_2=-q,\\
G_m &= (1+q^{m-1})G_{m-1}+qG_{m-2}-qG_{m-3}, \qquad G_0=1,\ G_1=1,\ G_2=1+q.
\end{aligned}$$

Proof. The identities A.59, A.60, and A.61 in slater's list are stated as follows.

Identity A.59 (Rogers [14]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q;q^2)_{n+1}(q;q)_n}=\frac{(q^2,q^{12},q^{14};q^{14})_\infty}{(q;q)_\infty}, \tag{3.15}$$

$$P_n=P_{n-1}+(q+q^n)P_{n-2}-qP_{n-3}, \qquad P_0=0,\ P_1=1,\ P_2=1. \tag{3.16}$$

Identity A.60 (Rogers [14]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q^2)_{n+1}(q;q)_n}=\frac{(q^4,q^{10},q^{14};q^{14})_\infty}{(q;q)_\infty}, \tag{3.17}$$

$$Q_n=Q_{n-1}+(q+q^n)Q_{n-2}-qQ_{n-3}, \qquad Q_0=1,\ Q_1=1,\ Q_2=1+q+q^2. \tag{3.18}$$

Identity A.61 (Rogers [13]):

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q^2)_{n}(q;q)_n}=\frac{(q^6,q^8,q^{14};q^{14})_\infty}{(q;q)_\infty}, \tag{3.19}$$

$$R_n=R_{n-1}+(q+q^n)R_{n-2}-qR_{n-3}, \qquad R_0=1,\ R_1=1+q,\ R_2=1+q+q^2. \tag{3.20}$$

The polynomials P_n, Q_n, and R_n converge to the right hand sides of (3.15), (3.17), and (3.19), respectively.

Consider the following determinant:

$$F(z):=\begin{vmatrix} 1 & q+zq & -q & & & \cdots \\ -1 & 1 & q+zq^2 & -q & & \cdots \\ & -1 & 1 & q+zq^3 & -q & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z)=F(zq)+(q+zq)F(zq^2)-qF(zq^3).$$

Setting

$$F(z)=\sum_{n=0}^{\infty} a_n z^n,$$

we get, upon comparing coefficients,

$$a_n = q^n a_n + q^{2n+1} a_n + q^{2n-1} a_{n-1} - q^{3n+1} a_n,$$

$$a_n = \frac{q^{2n-1}}{(1-q^{2n+1})(1-q^n)} a_{n-1} = \cdots = \frac{q^{n^2}(1-q)}{(q;q^2)_{n+1}(q;q)_n} a_0.$$

Since $a_0 = \frac{1}{1-q}$, we have

$$F(z)=\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q^2)_{n+1}(q;q)_n} z^n.$$

Thus we get

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q^2)_{n+1}(q;q)_n} = F(q^m), \tag{3.21}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q^2)_{n}(q;q)_n} = F(q^m) - qF(q^{m+2}). \tag{3.22}$$

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1 & q+zq & -q & & \cdots \\ -1 & 1 & q+zq^2 & -q & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ & & -1 & 1 & q+zq^{n-1} \\ & & & -1 & 1 \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = D_{n-1}(z) + (q+zq^{n-1})D_{n-2}(z) - qD_{n-3}(z),$$
$$D_0(z)=1,\ D_1(z)=1,\ D_2(z)=1+q+zq.$$

Then we have

$$D_{n-m+1}(q^m) = D_{n-m}(q^m) + (q+q^n)D_{n-m-1}(q^m) - qD_{n-m-2}(q^m).$$

Since $\langle D_{n-m+1}(q^m)\rangle_n$, $\langle P_n\rangle_n$, $\langle Q_n\rangle_n$, and $\langle R_n\rangle_n$ satisfy the same recursion, we set

$$D_{n-m+1}(q^m) = \lambda_m P_n + \mu_m Q_n + \nu_m R_n.$$

Using the initial conditions $D_0(q^m)=1$, $D_1(q^m)=1$, and $D_2(q^m)=1+q+q^{m+1}$, we have

$$\lambda_m = \frac{\begin{vmatrix} 1 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \\ 1+q+q^{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\mu_m = \frac{\begin{vmatrix} P_{m-1} & 1 & R_{m-1} \\ P_m & 1 & R_m \\ P_{m+1} & 1+q+q^{m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\nu_m = \frac{\begin{vmatrix} P_{m-1} & Q_{m-1} & 1 \\ P_m & Q_m & 1 \\ P_{m+1} & Q_{m+1} & 1+q+q^{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

where

$$\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix} = (-1)^{m-1} q^m,$$

which can be proved by induction on m. Therefore, we have simpler forms for λ_m, μ_m, and ν_m as follows:

$$\lambda_m = \frac{(-1)^m}{q^{m-1}} \begin{vmatrix} 0 & Q_{m-2} & R_{m-2} \\ 1 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \end{vmatrix},$$

$$\mu_m = \frac{(-1)^m}{q^{m-1}} \begin{vmatrix} P_{m-2} & 0 & R_{m-2} \\ P_{m-1} & 1 & R_{m-1} \\ P_m & 1 & R_m \end{vmatrix},$$

$$\nu_m = \frac{(-1)^m}{q^{m-1}} \begin{vmatrix} P_{m-2} & Q_{m-2} & 0 \\ P_{m-1} & Q_{m-1} & 1 \\ P_m & Q_m & 1 \end{vmatrix}.$$

According to (3.21) and (3.22), by setting

$$\begin{cases} E_m = \lambda_m - q\lambda_{m+2}, \\ F_m = \mu_m - q\mu_{m+2}, \\ G_m = \nu_m - q\nu_{m+2}, \end{cases}$$

we have

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q^2)_{n+1}(q;q)_n} = \lambda_m P_\infty + \mu_m Q_\infty + \nu_m R_\infty,$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q;q^2)_n(q;q)_n} = E_m P_\infty + F_m Q_\infty + G_m R_\infty.$$

Letting the last two columns in the determinants of λ_{m-1}, λ_{m-2}, λ_{m-3} be the same as those of λ_m, we find a linear equation

$$\lambda_m = (1+q^{m-3})\lambda_{m-1} + q^{-1}\lambda_{m-2} - q^{-1}\lambda_{m-3}.$$

Using the initial conditions of P_n, Q_n, and R_n, we have

$$\lambda_0 = q, \ \lambda_1 = 0, \ \lambda_2 = 1.$$

Proceeding in the same way, we get the recursions of μ_m, ν_m, E_m, F_m, and G_m. Therefore, we obtain (3.13) and (3.14). □

The identities (3.15) and (3.17) are the special cases of (3.13), and the identity (3.19) is a special case of (3.14).

Theorem 3.3. *We have*

(1)

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2m+1)/2}}{(q;q^2)_{n+1}(q;q)_n} = \frac{q^{-m}(q^2,q^5,q^7;q^7)_\infty (q^3,q^{11};q^{14})_\infty (-q;q)_\infty}{(q;q)_\infty} A_m$$

$$+\frac{q^{-m}(q,q^6,q^7;q^7)_\infty(q^5,q^9;q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}B_m$$
$$+\frac{q^{-m}(q^3,q^4,q^7;q^7)_\infty(q,q^{13};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}C_m, \tag{3.23}$$

where

$$A_m = qA_{m-1}+(q+q^{m-1})A_{m-2}-q^2A_{m-3}, \qquad A_0=1,\ A_1=0,\ A_2=q,$$
$$B_m = qB_{m-1}+(q+q^{m-1})B_{m-2}-q^2B_{m-3}, \qquad B_0=0,\ B_1=0,\ B_2=-q,$$
$$C_m = qC_{m-1}+(q+q^{m-1})C_{m-2}-q^2C_{m-3}, \qquad C_0=0,\ C_1=q,\ C_2=0.$$

(2)

$$\sum_{n=0}^{\infty}\frac{q^{n(n+2m+1)/2}}{(q;q^2)_n(q;q)_n}=\frac{(q^2,q^5,q^7;q^7)_\infty(q^3,q^{11};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}E_m$$
$$+\frac{(q,q^6,q^7;q^7)_\infty(q^5,q^9;q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}F_m$$
$$+\frac{(q^3,q^4,q^7;q^7)_\infty(q,q^{13};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}G_m, \tag{3.24}$$

where

$$E_m = E_{m-1}+(q+q^{m-1})E_{m-2}-qE_{m-3}, \qquad E_0=0,\ E_1=0,\ E_2=-q,$$
$$F_m = F_{m-1}+(q+q^{m-1})F_{m-2}-qF_{m-3}, \qquad F_0=1,\ F_1=1,\ F_2=1+q,$$
$$G_m = G_{m-1}+(q+q^{m-1})G_{m-2}-qG_{m-3}, \qquad G_0=0,\ G_1=-q,\ G_2=-q.$$

Proof. The identities A.80, A.81, and A.82 are stated as follows.

Identity A.80 (Rogers [14]):

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_{n+1}(q;q)_n}=\frac{(q^2,q^5,q^7;q^7)_\infty(q^3,q^{11};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}, \tag{3.25}$$

$$P_n=(1+q^n)P_{n-1}+qP_{n-2}-qP_{n-3}, \qquad P_0=1,\ P_1=1+q,\ P_2=1+2q+q^2+q^3. \tag{3.26}$$

Identity A.81 (Rogers [14]):

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q^2)_n(q;q)_n}=\frac{(q,q^6,q^7;q^7)_\infty(q^5,q^9;q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}, \tag{3.27}$$

$$Q_n=(1+q^n)Q_{n-1}+qQ_{n-2}-qQ_{n-3}, \qquad Q_0=1,\ Q_1=1+q,\ Q_2=1+q+q^2+q^3. \tag{3.28}$$

Identity A.82 (Rogers [14]):

$$\sum_{n=0}^{\infty}\frac{q^{n(n+3)/2}}{(q;q^2)_{n+1}(q;q)_n}=\frac{(q^3,q^4,q^7;q^7)_\infty(q,q^{13};q^{14})_\infty(-q;q)_\infty}{(q;q)_\infty}, \tag{3.29}$$

$$R_n=(1+q^n)R_{n-1}+qR_{n-2}-qR_{n-3}, \qquad R_0=0,\ R_1=1,\ R_2=1+q^2. \tag{3.30}$$

The polynomials P_n, Q_n, and R_n converge to the right hand sides of (3.25), (3.27), and (3.29), respectively.

Consider the following determinant:

$$F(z):=\begin{vmatrix} 1+zq & q & -q & & & \cdots \\ -1 & 1+zq^2 & q & -q & & \cdots \\ & -1 & 1+zq^3 & q & -q & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z) = (1 + zq)F(zq) + qF(zq^2) - qF(zq^3).$$

Setting

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we get, upon comparing coefficients,

$$a_n = q^n a_n + q^n a_{n-1} + q^{2n+1} a_n - q^{3n+1} a_n,$$

$$a_n = \frac{q^n}{(1 - q^{2n+1})(1 - q^n)} a_{n-1} = \cdots = \frac{q^{(n^2+n)/2}(1 - q)}{(q; q^2)_{n+1}(q; q)_n} a_0.$$

Since $a_0 = \frac{1}{1-q}$, we have

$$F(z) = \sum_{n=0}^{\infty} \frac{q^{(n^2+n)/2}}{(q; q^2)_{n+1}(q; q)_n} z^n.$$

Thus we get

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2m+1)/2}}{(q; q^2)_{n+1}(q; q)_n} = F(q^m), \tag{3.31}$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2m+1)/2}}{(q; q^2)_n(q; q)_n} = F(q^m) - qF(q^{m+2}). \tag{3.32}$$

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+zq & q & -q & & \cdots \\ -1 & 1+zq^2 & q & -q & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ & & -1 & 1+zq^{n-1} & q \\ & & & -1 & 1+zq^n \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1 + zq^n)D_{n-1}(z) + qD_{n-2}(z) - qD_{n-3}(z),$$
$$D_0(z) = 1,\ D_1(z) = 1 + zq,\ D_2(z) = 1 + q + zq + zq^2 + z^2q^3.$$

Then we have

$$D_{n-m}(q^m) = (1 + q^n)D_{n-m-1}(q^m) + qD_{n-m-2}(q^m) - qD_{n-m-3}(q^m).$$

Since $\langle D_{n-m}(q^m)\rangle_n$, $\langle P_n\rangle_n$, $\langle Q_n\rangle_n$, and $\langle R_n\rangle_n$ satisfy the same recursion, we set

$$D_{n-m}(q^m) = \lambda_m P_n + \mu_m Q_n + \nu_m R_n.$$

Using the initial conditions $D_0(q^m) = 1$, $D_1(q^m) = 1+q^{m+1}$, and $D_2(q^m) = 1+q+q^{m+1}+q^{m+2}+q^{2m+3}$, we have

$$\lambda_m = \frac{\begin{vmatrix} 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \\ 1+q^{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\mu_m = \frac{\begin{vmatrix} P_{m-1} & 0 & R_{m-1} \\ P_m & 1 & R_m \\ P_{m+1} & 1+q^{m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\nu_m = \frac{\begin{vmatrix} P_{m-1} & Q_{m-1} & 0 \\ P_m & Q_m & 1 \\ P_{m+1} & Q_{m+1} & 1+q^{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

where

$$\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix} = (-1)^{m-1} q^m,$$

which can be proved by induction on m. Therefore, we have simpler forms for λ_m, μ_m, and ν_m as follows:

$$\lambda_m = \frac{(-1)^m}{q^{m-1}} \begin{vmatrix} 0 & Q_{m-2} & R_{m-2} \\ 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \end{vmatrix},$$

$$\mu_m = \frac{(-1)^m}{q^{m-1}} \begin{vmatrix} P_{m-2} & 0 & R_{m-2} \\ P_{m-1} & 0 & R_{m-1} \\ P_m & 1 & R_m \end{vmatrix},$$

$$\nu_m = \frac{(-1)^m}{q^{m-1}} \begin{vmatrix} P_{m-2} & Q_{m-2} & 0 \\ P_{m-1} & Q_{m-1} & 0 \\ P_m & Q_m & 1 \end{vmatrix}.$$

According to (3.31) and (3.32), by setting

$$\begin{cases} A_m = q^m \lambda_m, \\ B_m = q^m \mu_m, \\ C_m = q^m \nu_m, \end{cases} \qquad \text{and} \qquad \begin{cases} E_m = \lambda_m - q\lambda_{m+2}, \\ F_m = \mu_m - q\mu_{m+2}, \\ G_m = \nu_m - q\nu_{m+2}, \end{cases}$$

we have

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2m+1)/2}}{(q;q^2)_{n+1}(q;q)_n} = q^{-m} A_m P_\infty + q^{-m} B_m Q_\infty + q^{-m} C_m R_\infty,$$

$$\sum_{n=0}^{\infty} \frac{q^{n(n+2m+1)/2}}{(q;q^2)_n(q;q)_n} = E_m P_\infty + F_m Q_\infty + G_m R_\infty.$$

Since

$$A_m = (-1)^m \begin{vmatrix} 0 & Q_{m-2} & R_{m-2} \\ 0 & Q_{m-1} & R_{m-1} \\ q & Q_m & R_m \end{vmatrix},$$

by letting the last two columns in the determinants of A_{m-1}, A_{m-2}, and A_{m-3} be the same as those of A_m, we find a linear equation

$$A_m = qA_{m-1} + (q + q^{m-1})A_{m-2} - q^2 A_{m-3}.$$

Using the initial conditions of P_n, Q_n, and R_n, we have

$$A_0 = 1,\ A_1 = 0,\ A_2 = q.$$

Proceeding in the same way, we get the recursions of B_m, C_m, E_m, F_m, and G_m. Therefore, we obtain (3.23) and (3.24). □

The identities (3.25) and (3.29) are the special cases of (3.23), and (3.27) is a special case of (3.24).

Theorem 3.4. *We have*

(1)

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}q^{n^2+2mn}}{(q;q^2)_{2n+1}} &= \frac{q^{-m}(q^3,q^{11},q^{14};q^{14})_\infty(q^8,q^{20};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}A_m \\ &+ \frac{q^{-m}(q,q^{13},q^{14};q^{14})_\infty(q^{12},q^{16};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}B_m \\ &+ \frac{q^{-m}(q^5,q^9,q^{14};q^{14})_\infty(q^4,q^{24};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}C_m, \qquad (3.33)\end{aligned}$$

where

$$\begin{aligned} A_m &= A_{m-1}+(q^2+q^{2m-4})A_{m-2}-q^2A_{m-3}, \qquad A_0=1,\ A_1=0,\ A_2=0,\\ B_m &= B_{m-1}+(q^2+q^{2m-4})B_{m-2}-q^2B_{m-3}, \qquad B_0=0,\ B_1=0,\ B_2=-q,\\ C_m &= C_{m-1}+(q^2+q^{2m-4})C_{m-2}-q^2C_{m-3}, \qquad C_0=q,\ C_1=q,\ C_2=q.\end{aligned}$$

(2)

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n}q^{n^2+2mn}}{(q;q^2)_{2n}} &= \frac{(q^3,q^{11},q^{14};q^{14})_\infty(q^8,q^{20};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}E_m \\ &+ \frac{(q,q^{13},q^{14};q^{14})_\infty(q^{12},q^{16};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}F_m \\ &+ \frac{(q^5,q^9,q^{14};q^{14})_\infty(q^4,q^{24};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}G_m, \qquad (3.34)\end{aligned}$$

where

$$\begin{aligned} E_m &= qE_{m-1}+(1+q^{2m-4})E_{m-2}-qE_{m-3}, \qquad E_0=1,\ E_1=0,\ E_2=1,\\ F_m &= qF_{m-1}+(1+q^{2m-4})F_{m-2}-qF_{m-3}, \qquad F_0=0,\ F_1=1,\ F_2=0,\\ G_m &= qG_{m-1}+(1+q^{2m-4})G_{m-2}-qG_{m-3}, \qquad G_0=0,\ G_1=0,\ G_2=-q.\end{aligned}$$

Proof. We give the identities A.117, A.118, and A.119 as follows.

Identity A.117 (Slater [18]):

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2}}{(q^2;q^2)_{2n}} = \frac{(q^3,q^{11},q^{14};q^{14})_\infty(q^8,q^{20};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \qquad (3.35)$$

$$\begin{aligned} P_n &= (1+q-q^2+q^{2n-1})P_{n-1}+(q^3+q^2-q)P_{n-2}-q^3P_{n-3},\\ &\quad P_0=1,\ P_1=1+q,\ P_2=1+q+q^2+q^4. \qquad (3.36)\end{aligned}$$

Identity A.118 (Slater [18]):

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2+2n}}{(q^2;q^2)_{2n}} = \frac{(q,q^{13},q^{14};q^{14})_\infty(q^{12},q^{16};q^{28})_\infty(-q;q^2)_\infty}{(q^2;q^2)_\infty}, \qquad (3.37)$$

$$\begin{aligned} Q_n &= (1+q-q^2+q^{2n-1})Q_{n-1}+(q^3+q^2-q)Q_{n-2}-q^3Q_{n-3},\\ &\quad Q_0=1,\ Q_1=1,\ Q_2=1+q^3. \qquad (3.38)\end{aligned}$$

Identity A.119 (Slater [18]):

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{n^2+2n}}{(q^2;q^2)_{2n+1}} = \frac{(q^5,q^9,q^{14};q^{14})_\infty (q^4,q^{24};q^{28})_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty}, \tag{3.39}$$

$$\begin{aligned} R_n &= (1+q-q^2+q^{2n-1})R_{n-1} + (q^3+q^2-q)R_{n-2} - q^3 R_{n-3}, \\ R_0 &= 0,\ R_1 = 1,\ R_2 = 1+q+q^3. \end{aligned} \tag{3.40}$$

The polynomials P_n, Q_n, and R_n converge to the right hand sides of (3.35), (3.37), and (3.39), respectively.

Consider the following determinant:

$$F(z) := \begin{vmatrix} 1+q-q^2+zq & q^3+q^2-q & -q^3 & & & \cdots \\ -1 & 1+q-q^2+zq^3 & q^3+q^2-q & -q^3 & & \cdots \\ & -1 & 1+q-q^2+zq^5 & q^3+q^2-q & -q^3 & \cdots \\ & & \ddots & \ddots & \ddots & \ddots \end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z) = (1+q-q^2+zq)F(zq^2) + (q^3+q^2-q)F(zq^4) - q^3 F(zq^6).$$

Setting

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

we get, upon comparing coefficients,

$$a_n = \frac{q^{2n-1}}{(1-q^{2n})(1-q^{2n+1})(1+q^{2n+2})} a_{n-1} = \cdots = \frac{q^{n^2}(1-q)(1+q^2)}{(q^2;q^2)_n (q;q^2)_{n+1} (-q^2;q^2)_{n+1}} a_0.$$

Since $a_0 = \frac{1}{(1-q)(1+q^2)}$, using some calculations of the q-shifted factorial, we have

$$F(z) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{n^2}}{(q^2;q^2)_{2n+1}(1+q^{2n+2})} z^n.$$

Thus we get

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{n^2+2mn}}{(q;q^2)_{2n+1}} = F(q^{2m}) + q^2 F(q^{2m+2}), \tag{3.41}$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q;q^2)_{2n}} = F(q^{2m}) + (q^2-q)F(q^{2m+2}) - q^3 F(q^{2m+4}). \tag{3.42}$$

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z) := \begin{vmatrix} 1+q-q^2+zq & q^3+q^2-q & -q^3 & & \cdots \\ -1 & 1+q-q^2+zq^3 & q^3+q^2-q & -q^3 & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ & & -1 & 1+q-q^2+zq^{2n-3} & q^3+q^2-q \\ & & & -1 & 1+q-q^2+zq^{2n-1} \end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$\begin{aligned} D_n(z) &= (1+q-q^2+zq^{2n-1})D_{n-1}(z) + (q^3+q^2-q)D_{n-2}(z) - q^3 D_{n-3}(z), \\ D_0(z) &= 1,\ D_1(z) = 1+q-q^2+zq, \\ D_2(z) &= 1+q-q^3+q^4+zq+zq^2+zq^4-zq^5+z^2q^4. \end{aligned}$$

Then we have

$$D_{n-m}(q^{2m}) = (1+q-q^2+q^{2n-1})D_{n-m-1}(q^{2m}) + (q^3+q^2-q)D_{n-m-2}(q^{2m}) - q^3 D_{n-m-3}(q^{2m}).$$

Since $\langle D_{n-m}(q^{2m})\rangle_n$, $\langle P_n\rangle_n$, $\langle Q_n\rangle_n$, and $\langle R_n\rangle_n$ satisfy the same recursion, we set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n + \nu_m R_n.$$

Using the initial conditions $D_{-1}(q^{2m}) = 0$, $D_0(q^{2m}) = 1$, and $D_1(q^{2m}) = 1 + q - q^2 + q^{2m+1}$, we have

$$\lambda_m = \frac{\begin{vmatrix} 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \\ 1+q-q^2+q^{2m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\mu_m = \frac{\begin{vmatrix} P_{m-1} & 0 & R_{m-1} \\ P_m & 1 & R_m \\ P_{m+1} & 1+q-q^2+q^{2m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\nu_m = \frac{\begin{vmatrix} P_{m-1} & Q_{m-1} & 0 \\ P_m & Q_m & 1 \\ P_{m+1} & Q_{m+1} & 1+q-q^2+q^{2m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

where

$$\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix} = (-1)^m q^{3m},$$

which can be proved by induction on m. Therefore, we have simpler forms for λ_m, μ_m, and ν_m as follows:

$$\lambda_m = \frac{(-1)^{m-1}}{q^{3m-3}} \begin{vmatrix} 0 & Q_{m-2} & R_{m-2} \\ 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \end{vmatrix},$$

$$\mu_m = \frac{(-1)^{m-1}}{q^{3m-3}} \begin{vmatrix} P_{m-2} & 0 & R_{m-2} \\ P_{m-1} & 0 & R_{m-1} \\ P_m & 1 & R_m \end{vmatrix},$$

$$\nu_m = \frac{(-1)^{m-1}}{q^{3m-3}} \begin{vmatrix} P_{m-2} & Q_{m-2} & 0 \\ P_{m-1} & Q_{m-1} & 0 \\ P_m & Q_m & 1 \end{vmatrix}.$$

According to (3.41) and (3.42), by setting

$$\begin{cases} A_m = q^m(\lambda_m + q^2\lambda_{m+1}), \\ B_m = q^m(\mu_m + q^2\mu_{m+1}), \\ C_m = q^m(\nu_m + q^2\nu_{m+1}), \end{cases} \quad \text{and} \quad \begin{cases} E_m = \lambda_m + (q^2-q)\lambda_{m+1} - q^3\lambda_{m+2}, \\ F_m = \mu_m + (q^2-q)\mu_{m+1} - q^3\mu_{m+2}, \\ G_m = \nu_m + (q^2-q)\nu_{m+1} - q^3\nu_{m+2}, \end{cases}$$

we have

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{n^2+2mn}}{(q;q^2)_{2n+1}} = q^{-m} A_m P_\infty + q^{-m} B_m Q_\infty + q^{-m} C_m R_\infty,$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q;q^2)_{2n}} = E_m P_\infty + F_m Q_\infty + G_m R_\infty.$$

Since

$$A_m = \frac{(-1)^{m-1}}{q^{2m-2}} \begin{vmatrix} -1 & qQ_{m-2} & qR_{m-2} \\ 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \end{vmatrix},$$

by letting the last two columns in the determinants of A_{m-1}, A_{m-2}, and A_{m-3} be the same as those of A_m, we find a linear equation

$$A_m = A_{m-1} + (q^2 + q^{2m-4})A_{m-2} - q^2 A_{m-3}.$$

Using the initial conditions of P_n, Q_n, and R_n, we have

$$A_0 = 1,\ A_1 = 0,\ A_2 = 0.$$

Proceeding in the same way, we get the recursions of B_m, C_m, E_m, F_m, and G_m. Therefore, we obtain (3.33) and (3.34). □

The identity (3.39) is a special case of (3.33), and (3.35) and (3.37) are the special cases of (3.34).

Theorem 3.5. *We have*

(1)

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2mn}}{(-q;q^2)_{n+1}(q^4;q^4)_n} = \frac{(-1)^m q^{-m}(-q^2,-q^3,q^5;q^5)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty} A_m + \frac{(-1)^m q^{-m}(q^{10};q^{10})_\infty (q^{20};q^{20})_\infty}{(q;q^2)_\infty (q^5;q^{20})_\infty (q^4;q^4)_\infty} B_m + \frac{(-1)^m q^{-m}(-q,-q^4,q^5;q^5)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty} C_m, \tag{3.43}$$

where

$$A_m = (1+q^{2m-4})A_{m-1} + (q^2+q^{2m-4})A_{m-2} - q^2 A_{m-3}, \qquad A_0 = 1,\ A_1 = 0,\ A_2 = 0,$$
$$B_m = (1+q^{2m-4})B_{m-1} + (q^2+q^{2m-4})B_{m-2} - q^2 B_{m-3}, \qquad B_0 = -q,\ B_1 = -q,\ B_2 = -q,$$
$$C_m = (1+q^{2m-4})C_{m-1} + (q^2+q^{2m-4})C_{m-2} - q^2 C_{m-3}, \qquad C_0 = 0,\ C_1 = 0,\ C_2 = q.$$

(2)

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2mn}}{(-q;q^2)_n (q^4;q^4)_n} = \frac{(-q^2,-q^3,q^5;q^5)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty} E_m + \frac{(q^{10};q^{10})_\infty (q^{20};q^{20})_\infty}{(q;q^2)_\infty (q^5;q^{20})_\infty (q^4;q^4)_\infty} F_m + \frac{(-q,-q^4,q^5;q^5)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty} G_m, \tag{3.44}$$

where

$$E_m = -(q+q^{2m-3})E_{m-1} + (1+q^{2m-4})E_{m-2} + qE_{m-3}, \qquad E_0 = 1,\ E_1 = 0,\ E_2 = 1,$$
$$F_m = -(q+q^{2m-3})F_{m-1} + (1+q^{2m-4})F_{m-2} + F_{m-3}, \qquad F_0 = 0,\ F_1 = 0,\ F_2 = 2q,$$
$$E_m = -(q+q^{2m-3})G_{m-1} + (1+q^{2m-4})G_{m-2} + qG_{m-3}, \qquad G_0 = 0,\ G_1 = 1,\ G_2 = -q.$$

Proof. The identity A.21 in Slater's list is stated as follows.

Identity A.21 (Slater [18]):

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q;q^2)_n (q^4;q^4)_n} = \frac{(-q^2,-q^3,q^5;q^5)_\infty (q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{3.45}$$

$$P_n = (1-q-q^2-q^{2n-1})P_{n-1} + (q+q^2-q^3+q^{2n-2})P_{n-2} + q^3 P_{n-3},$$
$$P_0 = 1,\ P_1 = 1-q,\ P_2 = 1-q+2q^2+q^4. \tag{3.46}$$

Recently, McLaughlin et al. and Bowman et al. found two new Rogers-Ramanujan type identities in [12] and [6], respectively.

An identity (McLaughlin et al. [12, Eq. (2.5)]):

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2+2n}}{(-q;q^2)_{n+1}(q^4;q^4)_n}=\frac{(q^{10};q^{10})_\infty(q^{20};q^{20})_\infty}{(q;q^2)_\infty(q^5;q^{20})_\infty(q^4;q^4)_\infty}. \tag{3.47}$$

An identity (Bowman et al. [6, Thm. 2.7]):

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2+2n}}{(-q;q^2)_n(q^4;q^4)_n}=\frac{(-q,-q^4,q^5;q^5)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty}. \tag{3.48}$$

We can see that (3.47) and (3.48) are partners to (3.45). Therefore, we have

$$\begin{aligned}Q_n&=(1-q-q^2-q^{2n-1})Q_{n-1}+(q+q^2-q^3+q^{2n-2})Q_{n-2}+q^3Q_{n-3},\\ &\qquad Q_0=0,\ Q_1=1,\ Q_2=1-q-q^3,\end{aligned} \tag{3.49}$$

$$\begin{aligned}R_n&=(1-q-q^2-q^{2n-1})R_{n-1}+(q+q^2-q^3+q^{2n-2})R_{n-2}+q^3R_{n-3},\\ &\qquad R_0=1,\ R_1=1,\ R_2=1-q^3,\end{aligned} \tag{3.50}$$

where P_n, Q_n, and R_n converge to the right hand sides of (3.45), (3.47), and (3.48), respectively. The initial conditions for Q_n and R_n are obtained in the following analysis.

Now we consider the following determinant:

$$F(z):=\begin{vmatrix}1-q-q^2-zq & q+q^2-q^3+zq^2 & q^3 & & & \cdots\\ -1 & 1-q-q^2-zq^3 & q+q^2-q^3+zq^4 & q^3 & & \cdots\\ & -1 & 1-q-q^2-zq^5 & q+q^2-q^3+zq^6 & q^3 & \cdots\\ & & \ddots & \ddots & \ddots & \ddots\end{vmatrix}.$$

Expanding the determinant with respect to the first column, we get

$$F(z)=(1-q-q^2-zq)F(zq^2)+(q+q^2-q^3+zq^2)F(zq^4)+q^3F(zq^6).$$

Setting

$$F(z)=\sum_{n=0}^{\infty}a_nz^n,$$

we get, upon comparing coefficients,

$$a_n=\frac{-(1-q^{2n-1})q^{2n-1}}{(1-q^{2n})(1+q^{2n+1})(1+q^{2n+2})}a_{n-1}=\cdots=\frac{(-1)^n(q;q^2)_nq^{n^2}(1+q)(1+q^2)}{(-q;q^2)_{n+1}(q^4;q^4)_n(1+q^{2n+2})}a_0.$$

Since $a_0=\frac{1}{(1+q)(1+q^2)}$, we have

$$F(z)=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(-q;q^2)_{n+1}(q^4;q^4)_n(1+q^{2n+2})}z^n.$$

Thus we get

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2+2mn}}{(-q;q^2)_{n+1}(q^4;q^4)_n}=F(q^{2m})+q^2F(q^{2m+2}), \tag{3.51}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2+2mn}}{(-q;q^2)_n(q^4;q^4)_n}=F(q^{2m})+(q^2+q)F(q^{2m+2})+q^3F(q^{2m+4}). \tag{3.52}$$

On the other hand, $F(z)$ is the limit of the finite determinant

$$D_n(z):=\begin{vmatrix}1-q-q^2-zq & q+q^2-q^3+zq^2 & q^3 & & \cdots\\ -1 & 1-q-q^2-zq^3 & q+q^2-q^3+zq^4 & q^3 & \cdots\\ \vdots & \ddots & \ddots & \ddots & \ddots\\ & & -1 & 1-q-q^2-zq^{2n-3} & q+q^2-q^3+zq^{2n-2}\\ & & & -1 & 1-q-q^2-zq^{2n-1}\end{vmatrix}.$$

Expanding this determinant with respect to the last row, we get

$$D_n(z) = (1 - q - q^2 - zq^{2n-1})D_{n-1}(z) + (q + q^2 - q^3 + zq^{2n-2})D_{n-2}(z) + q^3 D_{n-3}(z),$$
$$D_{-1}(z) = 0,\ D_0(z) = 1,\ D_1(z) = 1 - q - q^2 - zq.$$

Then we have

$$D_{n-m}(q^{2m}) = (1 - q - q^2 - q^{2n-1})D_{n-m-1}(q^{2m}) + (q + q^2 - q^3 + q^{2n-2})D_{n-m-2}(q^{2m}) + q^3 D_{n-m-3}(q^{2m}).$$

Now we calculate the initial conditions of Q_n and R_n in (3.49) and (3.50). According to (3.51) and (3.52), we have

$$\begin{aligned} Q_\infty &= F(q^2) + q^2 F(q^4), \\ R_\infty &= F(q^2) + (q^2 + q)F(q^4) + q^3 F(q^6). \end{aligned}$$

Due to $\lim_{n\to\infty} D_{n-m}(q^{2m}) = F(q^{2m})$, we have

$$\begin{aligned} Q_n &= D_{n-1}(q^2) + q^2 D_{n-2}(q^4), \\ R_n &= D_{n-1}(q^2) + (q^2 + q)D_{n-2}(q^4) + q^3 D_{n-3}(q^6). \end{aligned}$$

Therefore, we get

$$\begin{aligned} &Q_0 = 0,\ Q_1 = 1,\ Q_2 = 1 - q - q^3; \\ &R_0 = 1,\ R_1 = 1,\ R_2 = 1 - q^3. \end{aligned}$$

Since $\langle D_{n-m}(q^{2m})\rangle_n$, $\langle P_n\rangle_n$, $\langle Q_n\rangle_n$, and $\langle R_n\rangle_n$ satisfy the same recursion, we set

$$D_{n-m}(q^{2m}) = \lambda_m P_n + \mu_m Q_n + \nu_m R_n.$$

Using the initial conditions $D_{-1}(q^{2m}) = 0$, $D_0(q^{2m}) = 1$, and $D_1(q^{2m}) = 1 - q - q^2 - q^{2m+1}$, we have

$$\lambda_m = \frac{\begin{vmatrix} 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \\ 1 - q - q^2 - q^{2m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\mu_m = \frac{\begin{vmatrix} P_{m-1} & 0 & R_{m-1} \\ P_m & 1 & R_m \\ P_{m+1} & 1 - q - q^2 - q^{2m+1} & R_{m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

$$\nu_m = \frac{\begin{vmatrix} P_{m-1} & Q_{m-1} & 0 \\ P_m & Q_m & 1 \\ P_{m+1} & Q_{m+1} & 1 - q - q^2 - q^{2m+1} \end{vmatrix}}{\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix}},$$

where

$$\begin{vmatrix} P_{m-1} & Q_{m-1} & R_{m-1} \\ P_m & Q_m & R_m \\ P_{m+1} & Q_{m+1} & R_{m+1} \end{vmatrix} = -q^{3m-1}(1 + q),$$

which can be proved by induction on m. Therefore, we have simpler forms for λ_m, μ_m, and ν_m as follows:

$$\lambda_m = -\frac{\begin{vmatrix} 0 & Q_{m-2} & R_{m-2} \\ 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \end{vmatrix}}{q^{3m-4}(1+q)},$$

$$\mu_m = -\frac{\begin{vmatrix} P_{m-2} & 0 & R_{m-2} \\ P_{m-1} & 0 & R_{m-1} \\ P_m & 1 & R_m \end{vmatrix}}{q^{3m-4}(1+q)},$$

$$\nu_m = -\frac{\begin{vmatrix} P_{m-2} & Q_{m-2} & 0 \\ P_{m-1} & Q_{m-1} & 0 \\ P_m & Q_m & 1 \end{vmatrix}}{q^{3m-4}(1+q)}.$$

According to (3.41) and (3.42), by setting

$$\begin{cases} A_m = (-1)^m q^m(\lambda_m + q^2\lambda_{m+1}), \\ B_m = (-1)^m q^m(\mu_m + q^2\mu_{m+1}), \\ C_m = (-1)^m q^m(\nu_m + q^2\nu_{m+1}), \end{cases} \quad \text{and} \quad \begin{cases} E_m = \lambda_m + (q^2+q)\lambda_{m+1} + q^3\lambda_{m+2}, \\ F_m = \mu_m + (q^2+q)\mu_{m+1} + q^3\mu_{m+2}, \\ G_m = \nu_m + (q^2+q)\nu_{m+1} + q^3\nu_{m+2}, \end{cases}$$

we have

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_{n+1} q^{n^2+2mn}}{(q;q^2)_{2n+1}} = (-1)^m q^{-m} A_m P_\infty + (-1)^m q^{-m} B_m Q_\infty + (-1)^m q^{-m} C_m R_\infty,$$

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2+2mn}}{(q;q^2)_{2n}} = E_m P_\infty + F_m Q_\infty + G_m R_\infty.$$

Since

$$A_m = \frac{(-1)^{m-1}}{q^{2m-3}(1+q)} \begin{vmatrix} 1 & qQ_{m-2} & qR_{m-2} \\ 0 & Q_{m-1} & R_{m-1} \\ 1 & Q_m & R_m \end{vmatrix},$$

we can find a linear equation

$$A_m = (1+q^{2m-4})A_{m-1} + (q^2+q^{2m-4})A_{m-2} - q^2 A_{m-3}.$$

Using the initial conditions of P_n, Q_n, and R_n, we have

$$A_0 = 1, \ A_1 = 0, \ A_2 = 0.$$

Proceeding in the same way, we get the recursions of B_m, C_m, E_m, F_m, and G_m. Therefore, we obtain (3.43) and (3.44). □

The identity (3.47) is a special case of (3.43), and the identities (3.45) and (3.48) are the special cases of (3.44).

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